Swarm field dynamics and functional morphogenesis

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A class of models with application to swarm behavior as well as many other types of complex systems is studied with an emphasis on analytic techniques and results. Special attention is given to the role played by fluctuations in determining the behavior of such systems. In particular it is suggested that such fluctuations may play an active role, and not just the usual passive one, in the organization of structure in the vicinity of a non-equilibrium phase transition. One model, that of an ant swarm, is analyzed in more detail as an illustration of these ideas.

1 Introduction

In recent years there has been a great deal of interest in complex adaptive systems. Typically such systems are composed of a number of fairly simple interacting components which give rise to some form of global cooperative behavior. Sometimes this global behavior is studied in its own right, and sometimes it is allowed to evolve under external selection pressures which act according to some evolutionary algorithm on the local rules. Often the source of order, or an ordering transition in such systems is due to direct couplings between the basic components. These models often correspond either directly, or to some generalization of Ising and Spin glass type models in statistical physics. Concepts such a frustration, multiple basins of attraction, and rugged fitness landscapes play an important role in the understanding of the complex behavior of such system.

In this paper a somewhat different approach is pursued which is in line with the basic principles acting behind a variety of systems in nature, particularly biological ones. Here we study what will be called **sigmergic processes**, or **statistical swarm fields**, to be defined as systems of autonomous agents, referred to as particles, which interact not via direct couplings, but through a medium of environmental morphogenetic fields. The morphogenetic fields both determine the behavior of the individual agents, and are in turn acted on by the agents.

There are many good reason why one would want to study such processes or **swarms**. Swarms are a useful paradigmatic example of a complex adaptive systems in which a number of simple components exhibit emergent behaviors on the collective scale. These emergent properties, when taken in the proper context such as an ant swarm, may represent biological functionality or "swarm intelligence". Such a study might also be expected to shed light how cooperative behavior evolved, and on the evolutionary distance between various types of swarm behavior, questions of profound interest to social biologists. Lastly such process allow an exploration of the relationship between non-equilibrium phase transitions and complex behavior in model systems of interacting autonomous agents.

In the view of the author, some of the recent approaches to the problem of the collective behavior of autonomous agents, and to artificial life in general, have several weaknesses. One of the purposes of this work is to address some of these inadequacies via the application of more rigorous techniques. Thus, in addition to obtaining analytic type results for a class of specific systems with applications to swarm behavior, one of the main goals of this paper is the introduction of a general set of techniques which can be used as a framework for the study of a great variety of such systems.

The root inadequacy of much of the work in this area is a general lack of emphasis on analytic results, and an over dependence on computer simulations. It is quite often the case that, even when an analytic tool is available for the study of a particular problem, many researchers are still content with simulation. This is the result of an emphasis on behavior alone, instead of families of behavior which should be the true object of study. *The ability to simulate complex behavior is not the same thing as the ability to understand it.* Although there can be little doubt as to the value of simulations, particularly as a laboratory for the exploration of the basic phenomena, simulations almost never provide certain types of information which are central to the most ambitious goals of artificial life. Analysis can often provide information about families of behavior which simulation is simply unable to provide, and it is precisely this information which is of central importance to a general understanding of complex systems. This is particularly the case when one seeks to understand any system upon which adaptation on the space of behaviors plays, or has played, an important role.

Another inadequacy which commonly besets both the conceptual and modeling process is the lack of sufficient appreciation of the role of fluctuations. It is interesting to note that even with the current emphasis on bottom-up modeling, there is still a very prevalent top-down lack of familiarity with some of the more profound effects of noise on complex systems. Fluctuations can in fact an play a constructive role in the creation of order and complex behavior going well beyond the passive initiation of order producing transitions via an amplification process,[1] to the active production of structure which has no deterministic analog,[2] a fact which is still not sufficiently well understood by many researchers. Most complex adaptive systems will fall into this category since fluctuations in such systems are often large, and usually enter in a nonlinear, multiplicative way. This is frequently the result of the necessary balancing act between the *constructive* poles of reliability and adaptability. Therefore, in this paper we emphasize the points in the modeling process where fluctuations enter into the description. This is made possible by the fact that we will start with a full theoretical formulation, which contains all the information about fluctuations we may wish to make use of. It is then possible to work backward toward the full theory through a sequence of successively better approximations which we hope will capture important characteristic of the full system.

Since there is not space here for a detailed exposition, many results have been outlined. References are made to the appropriate sources for those who wish to see a full exposition.

2 Stigmergic Processes

2.1 Basic ingredients

Stigmergic processes are here introduced as a generalization of the concept of stigmergy (from the Greek meaning "incite to work") introduced by Grassé[3] as a hypothesis about the collective nest building of social insects. The hypothesis of stigmergy, as described by Wilson,[4] is that *it is the work already accomplished, rather than direct communication among nest mates, that induces the insects to perform additional labor.* The concept of stigmergy has also been invoked more recently in regards to swarm behavior.[5]

The more generalized idea of a stigmergic process is realized here in systems composed of three basic ingredients. The first ingredient is a *particle dynamics*. The particles represent the autonomous agents or organism, and the dynamics describes how they move in space. Since we are dealing with very unreliable components, this dynamics is necessarily statistical in description. The second ingredient is the *morphogenetic field dynamics*. There will be a field, or number of fields describing the environment which the organisms sense, and in turn influence by their actions. The dynamics of the morphogenetic fields describes how the environment changes with time. Lastly, some form of *coupling* is made between the particles and the fields. This coupling takes the form of a behavioral function which tells the particle how to move in response to the morphogenetic fields, and in turn, how the particles act back on these fields. The stigmergic element is imposed by the simplification that the particles are not influenced by each other via direct couplings, but only through the environment.

2.2 Particle dynamics

The dynamics of a *particle* is a Markov process on some finite state space $\mathcal{X}$, which may be either continuous or discrete. The particle density $\rho(\mathbf{x},\tau)$ obeys the Master equation

$$\frac{\partial \rho(\mathbf{x},\tau)}{\partial \tau} = \int_{\mathcal{X}} \{W_\tau(\mathbf{x}|\mathbf{y})\rho(\mathbf{y},\tau) - W_\tau(\mathbf{y}|\mathbf{x})\rho(\mathbf{x},\tau)\}\, d^D\mathbf{y}, \qquad (1)$$

where $W_\tau(\mathbf{x}|\mathbf{y})$ is the probability density to go from state $\mathbf{y}$ to $\mathbf{x}$ at time τ. All of the statistical properties of the particle flow can be calculated in theory from the master equation, and we will assume for our purposes that the fluctuations in the system are primarily driven by these statistical properties, which are due to the discreteness of the particles, and random nature of their motion. As we shall see, our description of the morphogenetic fields will not include intrinsic fluctuations, which are expected to be of negligible importance when compared to the fluctuations of the particle density. This is because such fields will general be composed of a vast number of components, for example, molecules of some pheromonal substance, so we can effectively model them as continuous valued quantities at every point in space. Note that this by does not mean that the morphogenetic fields will not fluctuate, only that these fluctuations will be driven by the particle fluctuations only.

Most generally we will be interested in open systems, where there is both a flow of particles into $\mathcal{X}$ from external sources, and a flow of particles out of the system

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[1] G. Nicolis, & I. Prigogine. *Self-Organization in Nonequilibrium Systems*, Wiley (1977).

[2] W. Horsthemke, & R. Lefever. *Noise Induced Transitions*, Springer-Verlag (1984).

[3] P. P. Grassé. *Experientia* 15, 356 (1959).

[4] E. O. Wilson. *The Insect Societies*, Belknap (1971).

[5] G. Theraulaz, & J.-L. Deneubourg. *SFI Working Paper* 92-09-046 (1992).

at various points. In such a case it is useful to introduce an extended space of states $\mathcal{X}^+ = \mathcal{X} + \mathbf{e}$. where $\mathbf{e}$ represents states external to $\mathcal{X}$. which play the role of a particle sink. In this case we can write

$$\frac{\partial \rho^+(\mathbf{x},\tau)}{\partial \tau} = \int_{\mathcal{X}^+} \{W_\tau(\mathbf{x}|\mathbf{y})\rho^+(\mathbf{y},\tau) \\ - W_\tau(\mathbf{y}|\mathbf{x})\rho^+(\mathbf{x},\tau)\} d^D\mathbf{y} + f_\tau(\mathbf{x}), \quad (2)$$

where $\rho^+(\mathbf{x},\tau) = \rho(\mathbf{x},\tau) + \rho_\mathbf{e}(\tau)\ \delta(\mathbf{x}-\mathbf{e})$ is composed of a continuous part $\rho(\mathbf{x},\tau)$ on $\mathcal{X}$, and a discrete part $\rho_\mathbf{e}(\tau)$ on $\mathbf{e}$. The inhomogeneous term $f_\tau(\mathbf{x})$ is the flow of density at $\mathbf{x}$ into $\mathcal{X}$ from an *outside* which we will regard as being entirely independent of the set of states $\mathbf{e}$.

As a consequence of the local conservation equation

$$\int_{\mathcal{X}} W_\tau(\mathbf{y}|\mathbf{x})\ d^D\mathbf{y} = 1, \quad (3)$$

the Master equation for a closed system can be put in the form

$$\frac{\partial \rho(\mathbf{x},\tau)}{\partial \tau} = \int_{\mathcal{X}} \mathcal{W}_\tau(\mathbf{x}|\mathbf{y})\rho(\mathbf{y},\tau)\ d^D\mathbf{y}. \quad (4)$$

Likewise as a consequence of

$$\int_{\mathcal{X}^+} W_\tau(\mathbf{y}|\mathbf{x})\ d^D\mathbf{y} = 1, \quad (5)$$

for open systems, the Master equation can be put in the form

$$\frac{\partial \rho^+(\mathbf{x},\tau)}{\partial \tau} = \int_{\mathcal{X}^+} \mathcal{W}_\tau(\mathbf{x}|\mathbf{y})\rho^+(\mathbf{y},\tau)\, d^D\mathbf{y} + f_\tau(\mathbf{x}), \quad (6)$$

where $\mathcal{W}_\tau(\mathbf{x}|\mathbf{y}) = W_\tau(\mathbf{x}|\mathbf{y}) - \delta(\mathbf{x}-\mathbf{y})$.

2.3 The morphogenetic fields

The second basic element of a Stigmergic process is a morphogenetic field $\sigma(\mathbf{x},\tau)$, or fields $\sigma_i(\mathbf{x},\tau)$ on $\mathcal{X}$. The morphogenetic fields represents the set of environmental stimuli which the particles respond to, and act on. In general the dynamics of these fields will be described by partial differential equations of the form $\partial_\tau\sigma = \Phi[\sigma,\rho;\pi_f]$, where π_f are a set of field parameters. A few examples will be mentioned as illustrations, but there are a nearly endless variety of both types and dynamics of the stimuli.

One of the simplest situations is a fixed one-component pheromonal field which evolves according to

$$\frac{\partial \sigma(\mathbf{x},\tau)}{\partial \tau} = -\kappa\ \sigma + \eta\ \rho, \quad (7)$$

where κ measures the rate of evaporation, breakdown or removal of the substance, and η the rate of emission of the pheromone by the organisms. In the case of a mobile pheromone. we can add a diffusion term

$$\frac{\partial \sigma(\mathbf{x},\tau)}{\partial \tau} = \mathcal{D}_\sigma \nabla^2\sigma - \kappa\ \sigma + \eta\ \rho, \quad (8)$$

where $\mathcal{D}_\sigma$ is a diffusion constant.

A more complicated situation could be envisioned in the case of nest building. Suppose we now have a two component field (σ_p,σ_n), where σ_p will be a pheromonal substance, and σ_n will be nest material. The nest material is picked up at a certain rate κ_n by the insects. In addition the material is laid down in proportion to some function $h(\sigma_n)$ of the amount of nest material present. We can write

$$\frac{\partial \sigma_n(\mathbf{x},\tau)}{\partial \tau} = -\kappa\ \sigma_n\ \rho_0 + \eta\ h(\sigma_n)\ \rho_1, \quad (9)$$

where ρ_0 is the density of workers not carrying nest material, and ρ_1 the densities of worker who are carrying nest material. In this case, it is the nest material itself which emits pheromone at a rate η_p, where we might write for the evolution of the pheromonal field

$$\frac{\partial \sigma_p(\mathbf{x},\tau)}{\partial \tau} = \mathcal{D}_{\sigma_p} \nabla^2\sigma_p - \kappa_p\ \sigma_p + \eta_p\ \sigma_n. \quad (10)$$

The concentration of the pheromone σ_p will in turn determine where the insects go, and hence the building behavior.

This last type of model illustrates an important point about stigmergy. Originally the concept was used to refer to behavior which is stimulated only by work already done, that is, by the nest material. For our purposes there is no practical distinction between the construction of nest structures, and the "construction" of a pheromonal field. They play equivalent roles in the theory. Thus, in the present generalization, the criticism of the concept of stigmergy that in many situations pheromonal stimuli contribute along with other environmental stimuli is entirely irrelevant. The basic processes are the same whether the organism respond to the nest or to the pheromonal field, or both. The models studied here will be pheromonal field models. Another criticism, that a stigmergic machine will in general be unable to shut down when the job is finished, is patently incorrect. A closer look at the structure of such processes reveals that such mechanisms are quite easily incorporated into the behaviors of a stigmergic process as defined above. This is because the behavioral states are phases separated by critical boundaries which depend on the global environmental state. When the environmental state changes, the behavioral response can pass through such a phase transition point and shut on or off in a very sharp manner.

Other types of systems which fall into this class are swimming bacteria and algae,[6] physical trail formation, the evolution of river networks,[7] diffusive transport in polymeric materials,[8] population distribution models, various types of fractal growth phenomena,[9] and developmental morphogenesis.[10] Obviously such models can take virtually limitless forms. Multicomponent particle systems can be considered in addition to multicomponent morphogenetic fields. We will limit ourselves here to systems composed of one type of particle and one type of morphogenetic field in order to emphasis the basic principles.

2.4 The behavioral coupling

The transition matrix $\mathcal{W}$ will depend on the morphogenetic fields. While the dynamics of these fields are somewhat fixed by the physical situation which is being modeled, the response of the particles to the field as encoded by the transition matrix might represent a considerable range of behaviors. As we shall see, small changes in the microscopic behaviors of the particles can result in large changes in the global behavior of the swarm, or particle field. This variability has significant implications not only for the behavioral *response* of the swarm to external stimuli, but also in the *evolution* of cooperative behavior. Wilson has remarked that an understanding of how this occurs would *constitute a technical breakthrough of exciting proportions, for it will then be possible, by artificially changing the probability matrices, to estimate the true amount of behavioral evolution required to go from [the behavior of] one species to ... that of another.*[11] He has further remarked that such large behavioral changes resulting from small changes in the individual dynamics would provide evidence that social behavior evolves at least as rapidly as morphology in social insects. This could provide an explanation why *behavioral diversity far outstrips morphological diversity at the level of species and higher taxonomic categories* in social insects.

We will be interested in local dependencies, that is, where the individual behavior is determined only by local stimuli. This models the somewhat inevitable fact that the individual organism in the swarm have little or no knowledge of the state of the swarm as a whole. The collective response of the swarm, on the other hand, is capable of a global integration of environmental information. In this sense the swarm at any given time can be in a state which represents a *nonlocal* integrated response to the external stimuli. One of the main goals of the present work is to show how it is possible for local individual behavior to lead to collective global behavior, and how this global behaviors is encoded by the local dynamics.

In addition to its dependence on the local morphogenetic field, the transition matrix will depend on one or more behavioral parameters π_p, referred to as particle parameters. These parameters will determine the response of the individuals to σ. If σ represents the *input*, the internal parameters represent the *programming* of the "stigmergic machine". These parameters might vary from individual to individual, particularly between members of different castes, but here we will study swarms composed of individuals with identical behavior. In addition, the behavioral parameters might change in response to external stimuli, or change on the evolutionary time scale. A determination of the dependence of the global behavior on the behavioral parameters, as well as on the swarm parameters such as the number of participants, is the goal of the methods used here. Once this dependence is known the range and types of behaviors which can be encoded by various classes of individual behaviors can be examined in a more or less comprehensive way.

3 Simplifications

3.1 Adiabatic elimination of particle modes

In general the particle dynamics will be inhomogenous in time since the morphogenetic fields will change in time. However, for many purposes we will want to make the approximation that the morphogenetic fields, and hence the transition matrix, changes very slowly on scales typical of the particle dynamics relaxation time. This may often be true quite generally. However, this approximation is often quite good *in the region of a non-equilibrium phase transition*, even for systems where typically the time scales are not well separated. This is because in the region the unstable modes of a systems will exhibit critical slowing down, and will relax on a time scale much longer than the time scale of the stable modes. Since the unstable modes can be viewed as fixed parameters which determine the quasi-stationary stable modes, the stable modes are said to be *slaved* to the unstable modes.[12] It is precisely these regions of transition which we will be

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[6] J. O. Kessler. *Comments Theoretical Biology* 1, 85 (1989).

[7] S. Kramer, & M. Marder. *Phys. Rev. Lett.* 68, 205 (1992).

[8] R. W. Cox, & D. S. Cohen. *J. Polymer Sci.* *B* 27, 589 (1989).

[9] T. Vicsek. *Fractal Growth Phenomena*, World Scientific (1989).

[10] J. E. Mittenthal. In: *Lectures in the Sciences of Complexity* (D. Stein, ed.), Addison-Wesley (1989).

[11] Wilson.

[12] H. Haken. *Synergetics*, Third Ed., Springer-Verlag (1983).

most interested in, since the ordering behavior of such systems is largely determined by these critical points.

In our case it is the particle modes which are the stable ones, and we will study the particle dynamics with a transition matrix which varies quasi-statically. For convenience we will work in the bra-ket notation where $\rho(\mathbf{x},\tau) = \langle \mathbf{x}|\rho(\tau)\rangle$, and $\mathcal{W}(\mathbf{x}|\mathbf{y}) = \langle \mathbf{x}|\mathcal{W}|\mathbf{y}\rangle$. The left and right eigenvectors of $\mathcal{W}$ are given by $\mathcal{W}|\rho_\lambda\rangle = \lambda\,|\rho_\lambda\rangle$ and $\langle\rho_\lambda|\mathcal{W} = \lambda\,\langle\rho_\lambda|$ respectively. The spectral decomposition in terms of the eigenvectors of the operator $\mathcal{W}$ is written as $\sum_\lambda |\rho_\lambda\rangle\,\langle\rho_\lambda| = 1$. Eq. 4 can then be written as

$$\frac{\partial}{\partial\tau}|\rho_\lambda(\tau)\rangle = \mathcal{W}|\rho_\lambda(\tau)\rangle, \tag{11}$$

which has solutions of the form $|\rho_\lambda(\tau)\rangle = e^{\lambda\tau}|\rho_\lambda\rangle$. We normalize the eigenvectors according to $\langle\rho_\lambda|\rho_\lambda\rangle = 1$. The local conservation equations insure that $\langle\rho_0|\mathbf{x}\rangle = 1$, and $\int_{\mathcal{X}}\langle\mathbf{x}|\rho_\lambda\rangle\; d^D\mathbf{x} = \delta_{0\lambda}$. We can do likewise for the homogeneous equation for an open system on extended state space $\mathcal{X}^+$,

$$\frac{\partial}{\partial\tau}|\rho^+_\lambda(\tau)\rangle = \mathcal{W}^+|\rho^+_\lambda(\tau)\rangle. \tag{12}$$

where $\mathcal{W}|\rho^+_\lambda\rangle = \lambda|\rho^+_\lambda\rangle$. In this case all of the particles eventually flow into the adsorbing state $\mathbf{e}$, so that $\langle\mathbf{x}|\rho_0\rangle = N\delta(\mathbf{x}-\mathbf{e})$.

The situation is, of course, more complicated when there is a driving flow of organism into the network, $f(\mathbf{x}) = \langle\mathbf{x}|f\rangle$, so that we have the inhomogeneous equation

$$\frac{\partial}{\partial\tau}|\rho^+(\tau)\rangle = \mathcal{W}^+|\rho^+(\tau)\rangle + |f\rangle. \tag{13}$$

We are only interested in the long time distributions on $\mathcal{X}$, which are independent of any initial conditions. The stationary states can be calculated,[13]

$$|\rho_s\rangle = \begin{cases} N|\rho_0\rangle & \text{closed} \\ \sum_{\lambda\neq 0}\frac{1}{\lambda}|\rho^+_\lambda\rangle\langle\rho^+_\lambda|f\rangle. & \text{open}\end{cases} \tag{14}$$

In addition all the statistical properties of the fluctuation about this value can be calculated from the master equation. To determine these fluctuations for open systems one must of course also specify the statistical properties of the driving term $|f\rangle$.

The quasi-stationary particle densities will depend on the morphogenetic fields over the entire space $\mathcal{X}$, and on the external forces. In addition, since the transition matrix will depend on the particle parameters $\boldsymbol{\pi}_p$, the quasi-stationary particle density will also depend on these parameters, $\rho_s[\boldsymbol{\sigma},\boldsymbol{\pi}_p](\mathbf{x},\tau)$. Let us consider a general set of morphogenetic field equations $\partial_\tau\boldsymbol{\sigma} = \Phi[\boldsymbol{\sigma},\rho;\boldsymbol{\pi}_f]$. The particle modes can be adiabatically eliminated from the picture as discussed above, where we obtain the set of stochastic partial differential order parameter equations

$$\frac{\partial\boldsymbol{\sigma}(\mathbf{x},\tau)}{\partial\tau} = \Phi[\boldsymbol{\sigma},\rho_s[\boldsymbol{\sigma}:\boldsymbol{\pi}_p] + g(\boldsymbol{\sigma},\boldsymbol{\pi}_p)\xi(t);\boldsymbol{\pi}_f], \tag{15}$$

where $g(\boldsymbol{\sigma},\boldsymbol{\pi}_p)$ is a function describing the fluctuations of the quasi-stationary particle density about its mean value, and $\xi(t)$ is gaussian white noise, $\langle\xi(t)\rangle = 0$, $\langle\xi(t)\xi(t'$ Since ρ_s will depend on both the global state of $\boldsymbol{\sigma}$, and on the global boundary conditions, this is a globally coupled set of equations for the evolution of the morphogenetic fields. *Slaving of the particle field therefore allows an explicitly coupled global dynamics to emerge from the strictly local interactions of the model*, providing a key to how a globally integrated response may emerge from a system of locally acting agents. This type of global coupling through separation of time scales could be expected to be of the greatest importance in self-organizing systems composed of a hierarchy of processes, each acting on a different characteristic time scale. In this paper we will focus on how just two of these levels interact.

The noise in the system is of the multiplicative type, since the strength of the fluctuations about the quasi-stationary particle density will in general also depend on $\boldsymbol{\sigma}$. In addition to amplifying an instability which exists in the absence of noise, this type of fluctuation can also produce transitions and ordered behavior in its own right. One of the consequence of this fact is that slaved particle field will constructively determine the self-organization properties of the systems *through its fluctuating properties*, as well as through quasi-stationary values. This is a fact which should be constantly be born in mind when studying such models.

In many cases it may be necessary to proceed with caution. The steps outlined above will hold when the fluctuations are *small enough*, that is, when $g(\boldsymbol{\sigma},\boldsymbol{\pi}_p)$ is small enough. In many systems in nature, particularly ones composed of very many parts, this assumption is quite valid. However, if the system of particles we are considering is a swarm, the fluctuations may not be small enough since the number of participants may be relatively few. A much more detailed discussion of these subjects will appear shortly.[14]

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[13] M. M. Millonas. In *ALIFE III* (in press, 1992a).

[14] M. M Millonas. *Phys. Rev. A* (in preparation, 1993).

3.2 Detailed balance and the thermodynamic analogy

For now we will consider the case where the transition matrix takes the form

$$W(\mathbf{x}|\mathbf{y}) \propto f\left(\sigma(\mathbf{x})\right) g(r), \tag{16}$$

where f is some weighting function describing the effect of the field σ on the motion of the particles, and $g(r)$ is a probability distribution of jumps of length $r = |\mathbf{x} - \mathbf{y}|$. When properly normalized the transition matrix is given by

$$W(\mathbf{x}|\mathbf{y}) = \frac{f\left(\sigma(\mathbf{x})\right) g(r)}{\int f\left(\sigma(\mathbf{y}+\mathbf{r})\right) g(r)\; d^D\mathbf{r}'}. \tag{17}$$

Transition matrices of this type obey the detailed balance relations

$$W(\mathbf{x}|\mathbf{y}) f(\sigma(\mathbf{y})) = W(\mathbf{y}|\mathbf{x}) f(\sigma(\mathbf{x})). \tag{18}$$

The property of detailed balance allows us to determine all the statistical properties of the quasi-stationary particle field.

An one-to-one analogy with a thermodynamic system with energy $U(\sigma(\mathbf{x}))$ and temperature $T = \beta^{-1}$ can be made if we set

$$f(\sigma(\mathbf{x})) = \exp\left(-\beta U(\sigma(\mathbf{x}))\right), \tag{19}$$

where any parameter T which affects f can be regarded as a temperature parameter if $f(\sigma(\mathbf{x}); T)$ scales like

$$f(\sigma(\mathbf{x}); \alpha\, T) = f^{-\alpha}(\sigma(\mathbf{x}); T). \tag{20}$$

Statistical quantities of interest can then be calculated from the one particle partition function

$$Z = \frac{1}{V}\int d^D\mathbf{x}\; \exp\left(-\beta U(\sigma(\mathbf{x}))\right) \tag{21}$$

according to the usual prescriptions, where V is the total volume of the state space $\mathcal{X}$. The N particle partition function is $Z_N = Z^N$.

Let us partially evaluate the partition function over the volume μ_ϵ of phase space with a given energy ϵ,

$$Z = \frac{1}{V}\int_{U\neq\epsilon} d^D\mathbf{x}\; \exp\left(-\beta U(\sigma(\mathbf{x}))\right) + \frac{\mu_\epsilon}{V}\exp(-\beta\epsilon). \tag{22}$$

The mean particle density in the energy state ϵ is given by

$$\langle\rho_\epsilon\rangle = \frac{\langle n_\epsilon\rangle}{\mu_\epsilon} = -\frac{1}{\beta\mu_\epsilon}\frac{\partial \ln Z_N}{\partial\epsilon} = \frac{N}{VZ}\exp(-\beta\epsilon). \tag{23}$$

To first order the fluctuations are given by the density dispersion of the particles in the ϵ energy state.

$$\langle(\Delta\rho_\epsilon)^2\rangle = \frac{1}{\beta^2\mu_\epsilon^2}\frac{\partial^2 \ln Z_N}{\partial\epsilon^2} = \frac{\langle\rho_\epsilon\rangle}{\mu_\epsilon}\left(1 - \frac{\mu_\epsilon}{N}\langle\rho_\epsilon\rangle\right) \tag{24}$$

The slaved particle field in energy state ϵ can then be represented to lowest order in the fluctuations by

$$\rho_\epsilon = \langle\rho_\epsilon\rangle + \sqrt{\langle(\Delta\rho_\epsilon)^2\rangle}\;\xi(t). \tag{25}$$

3.3 Fixed pheromonal field

We will now consider in greater detail one of the simplest cases which has many applications in the area swarm dynamics, a fixed pheromonal field obeying the Eq. 7. The field parameters are the decay rate κ, and the emission rate η.

We will first illustrate the types of self-organization possible by such a morphogenetic field, when driven by a closed system of particles obeying detailed balance. After adiabatic elimination of the particle density the stationary condition on the pheromonal field can be written $\sigma(\mathbf{y}) f(\sigma(\mathbf{x})) = \sigma(\mathbf{x}) f(\sigma(\mathbf{y}))$.[15] Let us arbitrarily pick two different points $\mathbf{x}$ and $\mathbf{y}$, and let $r = \sigma_< / \sigma_>$ be the ratio of the pheromone densities at these points where $\sigma_< \leq \sigma_>$, so that $0 < r \leq 1$. We can write the stationary conditions as

$$g(\bar{r}) \equiv f(\bar{r}\;\sigma_>) - \bar{r}\; f(\sigma_>) = 0 \tag{26}$$

We want to inquire about the roots $\bar{r}$ the equation $g(\bar{r}) = 0$, that is, what are the number of possible ratios of pheromone density at differing point of the stationary distribution. One solution is always $\bar{r} = 1$, which corresponds to the homogeneous distribution. The homogeneous distribution gives rise to a new branch of solutions when $g'(1) = 0$, that is, at the point where the slope of $g(r)$ at $r = 1$ changes signs. The homogeneous state is unstable when

$$\chi(\gamma) > \frac{T}{\gamma}, \tag{27}$$

where $\chi(\sigma) = -U'(\sigma)$ is known as the chemotactic factor, which it analogous to a force, and γ is the mean value of σ, where $\gamma = N\eta/V\kappa$. Thus, for small values of the field density the field must be spatially homogeneous, $\sigma(\mathbf{x}) = \gamma$, and at a critical value of the mean field density this homogeneous solution becomes unstable. In general there will be a broken symmetry solution $\bar{r} < 1$. These solutions correspond to situations where the pheromonal field is in a bimodal state where it can take one of two possible values at each point.

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[15] Ibid.

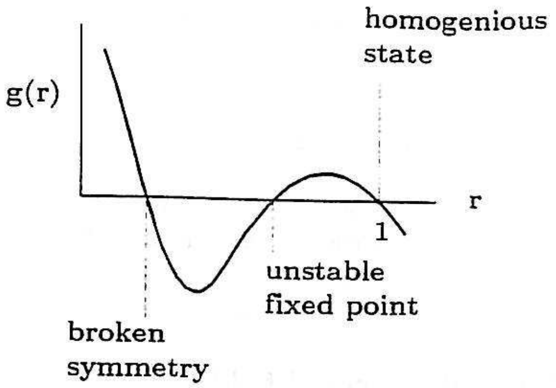

Figure 1: Typical stabiltiy curve g(r)

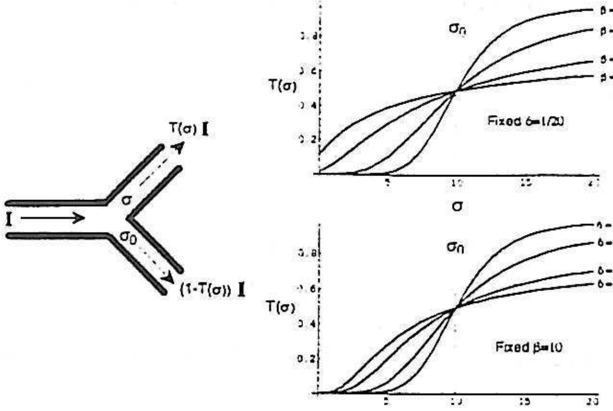

Figure 2: Transiton functions for varying β and δ.

The mean field order parameter equation for the fixed pheromonal field is

$$\frac{\partial \sigma(\mathbf{x}, \tau)}{\partial \tau} = -\kappa\, \sigma + \eta\, \langle \rho(\sigma) \rangle$$

$$= -\kappa\, \sigma + \frac{\eta N f(\sigma)}{\int d^D\mathbf{x}'\, f(\sigma(\mathbf{x}'))}. \tag{28}$$

The bimodal field values can be determined from Eq. 28 by noting that

$$\frac{1}{V} \int f(\sigma(\mathbf{x}))\, d^D\mathbf{x} = \mu^+ f(\sigma^+) + \mu^- f(\sigma^-), \tag{29}$$

where μ^+ and μ^- are the fractions of the total volume in state space which take the field values σ^+ and σ^- respectively. The stationary values are given by the solutions of the fixed point equations

$$\sigma^+ = \gamma(1+\upsilon)\left[1 + \upsilon / R(\sigma^+, \sigma^-)\right]^{-1}$$

$$\sigma^- = \gamma(1+\upsilon)\left[\upsilon + R(\sigma^+, \sigma^-)\right]^{-1}, \tag{30}$$

where $R(\mathbf{x}, \mathbf{y}) = f(\sigma(\mathbf{x}))/f(\sigma(\mathbf{y}))$, and $\upsilon = \mu^-/\mu^+$. These lead to an implicit equation for R,

$$0 = \Phi(\bar{R}; \gamma, \upsilon, \boldsymbol{\pi}_p)$$

$$\equiv f\left(\frac{\gamma(1+\upsilon)}{\upsilon + \bar{R}}; \boldsymbol{\pi}_p\right) \bar{R} - f\left(\frac{\gamma(1+\upsilon)}{1 + \upsilon/\bar{R}}; \boldsymbol{\pi}_p\right). \tag{31}$$

The roots $\bar{R}$ of Φ can then in principle be found as functions of γ, υ, and the particle parameters $\boldsymbol{\pi}_p$ which influence the transition function f.

In addition it can be shown[16] that if f is strictly excitatory ($f' \geq 0$) the function

$$\Psi(R; \sigma_0, \upsilon, \pi) \equiv \int_0^R dx\; \Phi(x; \sigma_0, \upsilon, \pi). \tag{32}$$

is bound from below,

$$\Psi(R; \sigma_0, \upsilon, \pi) \geq -\frac{f^2\left(\sigma_0(1+\upsilon); \pi\right)}{2 f(0; \pi)} \tag{33}$$

strictly decreasing $\dot{\Phi} < 0$. Ψ is therefore a Lyapunov function for the pheromonal field. Stable fixed points $\bar{R}$ are determined from $\Phi'(\bar{R}) = 0$ and $\Phi''(\bar{R}) > 0$, and critical points by $\Phi'(\bar{R}; \pi_*) = 0$ and $\Phi''(\bar{R}; \pi_*) = 0$.

4 Ant Swarms

4.1 Microscopic ant behavior

The microscopic dynamics of ants can be described by the pheromone energy function

$$U(\sigma) = -\ln\left(1 + \frac{\sigma}{1 + \delta\sigma}\right), \tag{34}$$

where σ is the pheromone density, and δ is a dimensionless behavioral parameter.[17] The function $f(\sigma)$ is given by Eq. 19, and the particle parameters are then δ and T. The temperature parameter $T = 1/\beta$ describes the internal randomness of the response of the ants to the pheromonal field. This function is based approximately on a model for Osmotropotaxi (scent gradient following),[18] and on experimental observations of actual ants.[19] For the case where the density of ants it low, and hence the pheromone density is low ($\sigma << 1/\delta$), we can make use of the approximate energy function $U_0(\sigma) = -\ln(1+\sigma)$. The constant $1/\delta$ will be known as the capacity. When σ approaches $1/\delta$ the ants respond less accurately to pheromone gradients.

An illustration of this effect is shown in Figure 2. A given current of organisms I flows into a junction from

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[16] Millonas (1992a).

[17] Millonas (1993).

[18] V. Calenbuhr, & J.-L. Deneubourg. *J. Theor. Biol.* **158**, 359 (1991).

[19] J.-L. Deneubourg. et. al. *J. Insect Behav.* **32** 159.

the left. On the lower branch the pheromone density is fixed at σ_0, and on the upper branch σ is allowed to vary. $\mathcal{T}(\sigma)$, the proportion of the current which flows into the upper branch, is given by the sigmoidal function

$$\mathcal{T}(\sigma) = [1 + \exp(\beta\epsilon(\sigma)/\epsilon(\sigma_0))]^{-1}. \tag{35}$$

The plots on the right of Figure 2 shows $\mathcal{T}(\sigma)$ for varying values of β and δ. The upper plot, where δ is fixed, shows the influence of increasing the temperature (lowering β). As the temperature increases the threshold response becomes less and less pronounced. In the opposite limit $\beta \to \infty$, $\mathcal{T}(\sigma)$ would be a step function $\Theta(\sigma - \sigma_0)$. In this limit all of the ants would choose the branch with the greatest pheromone density. In the lower plot the noise level is fixed, and the capacity $1/\delta$ is varied. It is interesting to note that the effects of decreasing the capacity with fixed temperature are similar to the effects of increasing the temperature with fixed capacity. When the density of the ants increases, the pheromone density increases up to and beyond the capacity, the qualitative effects on the behavior of the ants is the same *as if the temperature was increased.* This gives the swarm roughly the ability to modulate its temperature by modulating its numbers.

This can be made more clear by defining an effective temperature factor $\theta(\sigma)$ through the relation $f(\sigma) = \exp(-\beta U_0(\sigma)/\theta(\sigma))$. $\theta(\sigma)$ roughly measures the effective change in temperature as a function of the pheromonal field when compared to the case where $\delta = 0$, which correspond to the energy function U_0. The effective temperature is then given by $\theta(\sigma)T$ where

$$\theta(\sigma) = \frac{\ln\left(1 + \frac{\sigma}{1+\delta\sigma}\right)}{\ln(1+\sigma)}. \tag{36}$$

Fig. 3 illustrates the increase in the effective temperature with increasing σ for three different values of δ. Since increasing the temperature tends to decrease stability, we might expect any organized behavior to breakdown when the number of participants grows too large. It is this ability or the swarm to self-modify its temperature which allows it, in a sense, to traverse its various phase transition boundaries. Such boundaries are of crucial importance in self-organization and emergent phenomena, and it has been proposed that the ability to self-organize at or near these boundaries is at the heart of adaptive, emergent biological systems.

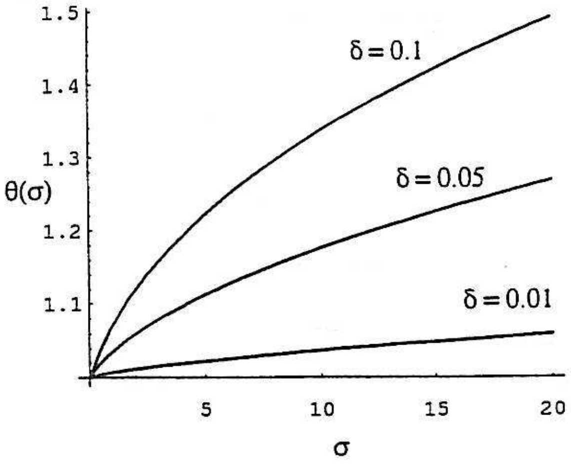

Figure 3: Effective temperature factor

4.2 Spontaneous symmetry breaking

The critical points of the homogeneous phase are given by the condition $\chi(\gamma_*) = T/\gamma_*$, which has solutions

$$\gamma_*^{\pm} = \frac{\beta - 2\delta - 1 \pm \sqrt{\beta^2 - 2\beta - 4\beta\delta + 1}}{2\delta(1+\delta)}, \tag{37}$$

where γ_*^- is the value of γ where the symmetric phase becomes unstable as γ is increased, and γ_*^+ is the value of γ where the symmetric phase again becomes stable. These critical points are independent of the order parameter v as could be expected, since in the homogeneous phase v does not really exist. These critical points themselves are the result of a bifurcation controlled by δ. This bifurcation only occurs for delta below the critical point

$$\delta < \delta_* = \frac{(1-T)^2}{4T}. \tag{38}$$

When $\delta > \delta_*$ no symmetry breaking is possible, irrespective of γ.

The broken symmetry states and critical points can be found from the Lyapunov function for the swarm

$$\begin{aligned}
\Psi = {} & \tfrac{R^{2+2\beta}}{2+2\beta}(1+\delta\gamma+\delta\gamma v) - \tfrac{R^{1+2\beta}}{1+2\beta}(1+\gamma+\delta\gamma+\gamma v+\delta\gamma v) \\
& + \tfrac{R^{2+\beta}}{2+\beta}(\gamma+\delta\gamma+\delta\gamma^2+\delta^2\gamma^2+2v+\gamma v+2\delta\gamma v+2\delta\gamma^2 v \\
& \qquad +2\delta^2\gamma^2 v+\delta\gamma v^2+\delta\gamma^2 v^2+\delta^2\gamma^2 v^2) \\
& - \tfrac{R^{1+\beta}}{1+\beta}(\delta\gamma+\delta\gamma^2+\delta^2\gamma^2+2v+\gamma v-2\delta\gamma v+2\delta\gamma^2 v+2\delta^2\gamma^2 v \\
& \qquad +\gamma v^2+\delta\gamma v^2+\delta\gamma^2 v^2+\delta^2\gamma^2 v^2) \\
& + \tfrac{R^2 v}{2}(\gamma+\delta\gamma+v+\gamma v+\delta\gamma v) - Rv(\delta\gamma+v+\delta\gamma v),
\end{aligned} \tag{39}$$

which is calculated as described above. There is a region of tristability, where either the inhomogeneous phase

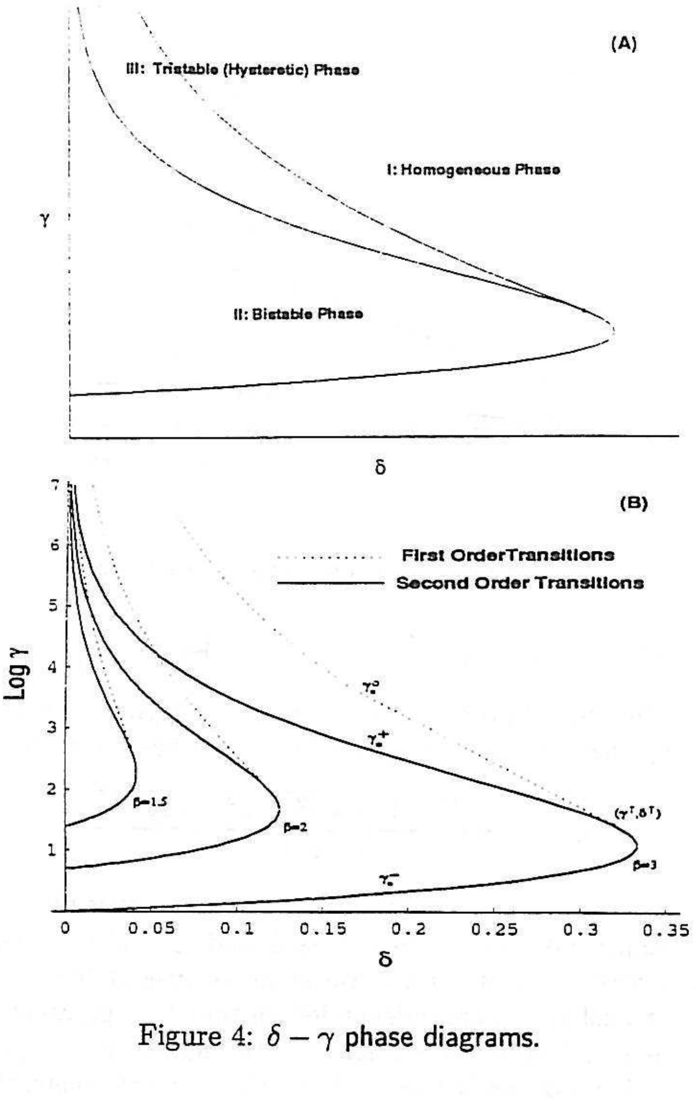

Figure 4: $\delta - \gamma$ phase diagrams.

or the homogeneous phase are possible. Which is chosen will depend on the initial conditions, and hysteresis (multiple values of the order parameter for the same values of the state variables) is possible. For a given δ this region extends from $\gamma_*^+ < \gamma < \gamma_*^0(v)$, where γ_*^0 mark the location of a first order transition. For certain case $\gamma_*^0(v)$ as a function of δ can be calculated analytically, but we will usually have to resort to Newton's method, or some other numerical scheme. All of this information can be illustrated by plotting the critical points $\gamma_*^\pm$ and $\gamma_*^0(1)$ as a functions of δ. The resulting phase diagram shown in Figure 4 illustrates the regions of symmetric phase, bistability and tristability. This plot is a simple illustration the effect of behavioral and swarm parameters on the swarm as a whole, and the relation these effects to each other. In this case δ is a behavioral parameter which could be expected to change on the evolutionary time scale, and γ, which is proportional to the number of participants, is a swarm parameter which determines the behavioral "phase" of the swarm.

In general the various inhomogeneous states labeled by v will become unstable at *different values of* γ. In this case we can have a quite complicated sequence of ordering transitions as γ is increased. In this regard the presence of the first order transition boundaries will play an important role. This is because after an initial symmetry breaking occurs upon passage through a second order transition, this broken symmetry state can itself become unstable *for non-zero value of the order parameter* M. These first order instabilities can produce a cascade, or hierarchy of ordering transitions. This ordering can be used to explore some other experiments with actual ants, but due to its complexity and the limited space here, whole subject is best taken up elsewhere.[20]

4.3 Order parameters and critical exponents

The effect of the slaving of the stable modes of a system at a critical point is to reduce the dimensionality of the systems down to an effective set of order parameter equations. There is not space here to fully discuss the complete reduction of the system, but the two order parameters we have introduced are R, and v. Instead of R we can make used of the easily observable order parameter $M = \rho_+ - \rho_-$, which is just the difference in the densities of ants on the nodes in the two states. In this regard this order parameter is similar to the order parameter of a gas-liquid transition, where ρ_+ would represent the density of the liquid and ρ^- the density of the gas. We can illustrate the phase transition at the critical temperature T_c and the resulting emergence of the order parameter m in Fig. 5(a). This behavior of the order parameter versus the temperature is reminiscent of equilibrium phase transitions in physics. The critical temperature is given, in terms of the other parameters as

$$T_c = \frac{\gamma}{1 + \gamma + 2\,\delta\,\gamma + \delta\,\gamma^2 + \delta^2\,\gamma^2}. \tag{40}$$

We can also plot the order parameter as a function of γ, that is, the equations of state, shown in Fig. 5(b), clearly illustrating both the second and first order transitions.

Very close to the critical points, the order parameter scales according to critical exponents which are independent of the particular parameters of the system. For instance, we obtain the mean-field critical behavior

$$M \sim |T - T_c|^{1/2}. \tag{41}$$

4.4 Swarms on a lattice

All of the above results hold when the configuration space is not continuous but a lattice. Such is the case in many of the laboratory experiments with actual ants

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[20] Millonas (1993).

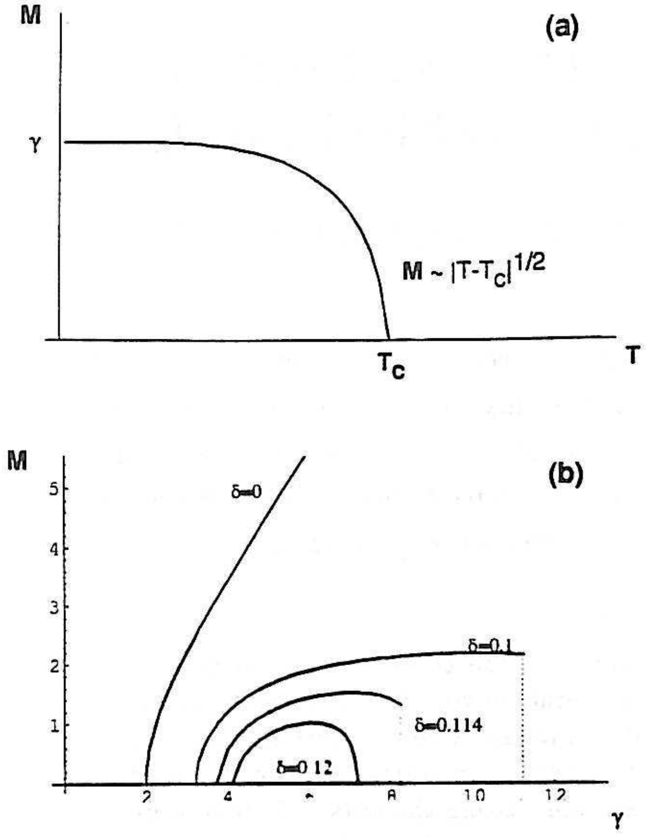

Figure 5: Plots of (a) the order parameter as a function of temperature, and (b) the equations of state for varying values of δ.

which have been recently been studied.[21] In these experiments ants are allowed to walk on bridges connected together in a somewhat arbitrary way, forming a network. Detailed balance is imposed by the experimental setup where at each "fork in the road" an ant can chose only between geometrically equivalent alternatives. The topology of the network is defined by a connection matrix, and the swarm can be described the discretized equations

$$\mu_i \frac{d\rho^i}{d\tau} = \sum_j v_0 (W_{ij}\rho^j - W_{ji}\rho^i) + f^i, \tag{42}$$

$$\frac{d\sigma^i}{d\tau} = -\kappa\, \sigma^i + \eta^i \rho^i. \tag{43}$$

where in this case different values of η^i in each segment are allowed. The particle densities are slaved to the pheromone densities via the quasi-stationary distribution

$$\rho^i = \frac{N \exp(-\beta U(\sigma^i))}{\sum_j \mu_j \, \exp(-\beta U(\sigma^j))}. \tag{44}$$

All of the previous result hold, except now the order parameter v is constrained to a discrete set of values $v = \sum \mu_{i^+} / \sum \mu_{i^-}$, where $\mu_{i\pm}$ are the lengths of the bridges in the ± mode. A discussion of how these equations lead to the various type of observed behaviors in the experimental situations has been published elsewhere.[22]

4.5 Incorporation of fluctuations

The results above show how the essentially infinite description for the pheromonal field is reduced to the bimodal field equations $\dot{\sigma}^{\pm} = -\kappa\, \sigma^{\pm} + \eta\, \rho^{\pm}$. Making use of the results at the end of section 3.2, this lead to a bimodal set of Langevin equations

$$\frac{d\sigma^{\pm}}{d\tau} = -\kappa\, \sigma^{\pm} + \eta\, \langle \rho^{\pm} \rangle + \sqrt{\langle (\Delta\rho^{\pm})^2 \rangle}\, \xi(t), \tag{45}$$

which leads to a Fokker-Planck equation

$$\begin{aligned} \partial_\tau \rho(\sigma^+, \sigma^-) &= \partial_{\sigma^+} \left((\kappa\, \sigma^+ - \langle \rho^+ \rangle) \rho \right) \\ &+ \partial_{\sigma^-} \left((\kappa\, \sigma^- - \langle \rho^- \rangle) \rho \right) \\ &+ \frac{1}{2} \left(\partial^2_{\sigma^+} (\langle (\Delta\rho^+)^2 \rangle \rho) + \partial^2_{\sigma^-} (\langle (\Delta\rho^-)^2 \rangle \rho) \right), \end{aligned} \tag{46}$$

where Eq. 45 has been interpreted in the Ito sense.

A more or less complete description incorporating the fluctuations is possible,[23] which we outline here. In the region of the non-equilibrium phase transition the stable mode $S = (\sigma^+ + v\, \sigma^-)/(1 + v)$ can be adiabatically eliminated from the picture, this time directly from Eq. 46. We can then write down a one dimension Fokker-Planck equation for the order parameter density $\rho(M)$ which has an analytic solution. The resulting solution allows the exploration of the effect of the fluctuations near the non-equilibrium phase transitions, and a treatment of the influence of fluctuations, not only on the critical exponents as in the case of equilibrium phase transitions, but also on the critical points. Since the critical points themselves will depend on the strength and type of fluctuations, *the fluctuations may play a active, creative role in the production of order in the vicinity of a non-equilibrium phase transition.*[24]

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[21] Deneubourg, et. al.

[22] M. M. Millonas. *J. Theor. Biol.* (in press,1992b); In: *Cooperative Phenomena and Cellular Automata* , (in press, 1992c).

[23] Millonas (1993).

[24] Millonas (1992).