# Theoretical and Practical Investigations of Lake Biopopulations

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A mathematical model of ecosystem of a lake-cooler of a nuclear power station is considered. A generalized case of an ecosystem simulation is formulated. The three component mathematical model is investigated with a help of the qualitative theory of differential equations. The conditions of stability are determined, the case of global stability is proved applying the Liapunov's functions. The practical model is based on the principle of matter conservation. It involves coefficients which depend on light intensity and temperature change. An ecological computing by numerical simulation allows to predict a behaviour and a life of different organisms, populations and communities in areas closed nuclear power stations. The ecological area considered in this paper is the natural lake as a coder of a station.

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A mathematical model of ecosystem of a lake-cooler of a nuclear power station is considered. A generalized case of an ecosystem simulation is formulated. The three component mathematical model is investigated with a help of the qualitative theory of differential equations. The conditions of stability are determined, the case of global stability is proved applying the Liapunov's functions. The practical model is based on the principle of matter conservation. It involves coefficients which depend on light intensity and temperature change. An ecological computing by numerical simulation allows to predict a behaviour and a life of different organisms, populations and communities in areas closed nuclear power stations. The ecological area considered in this paper is the natural lake as a coder of a station.

## 1. Introduction

One of the great ecological problems is security of nuclear power stations. In order to study all the aspects of ecological monitoring of a nuclear power stations one must solve the problems of control, regulation and management, to predict the behavioristic effects of a bio-community.

The analysis of the state of the lake Drukshiai in Lithuania, as a cooler of atomic station, has shown that very complicated trophic interactions are taking place. In spite of the fact that many mathematical models of ecosystems have been suggested or even partially realized, a lot of problems still exist, the majority of which concern the questions of stability, adaptivity and correspondence of the models to the actual behavioristic characteristics (Garliauskas, 1985; Garliauskas et al., 1987; Medvedev, 1970; Pykh, 1983; Volterra, 1976).

Taking into account an exceptional importance of examining the whole complex of the monitoring system and particularly approaching the problem systematically, we shall deal with a concrete mathematical model of a lake-cooler ecosystem, taking into consideration the thermal influence on the bio-community of the lake.

## 2. Three-component System

Let us examine a three-component biological community. The linkages between the components are the following:

$$A \rightarrow B \rightarrow C,$$

where the component with biomass $A$ is a victim as regards to the component with biomass $B$, and the component with biomass $B$ is a predatory as regards to the component with biomass $A$ and a victim as regards to the component with biomass $C$.

On the limits of the Volterra theory (Volterra, 1976) a model of this community can be represented by the following differential equations:

$$\begin{cases} \dot{x}_1 = A_1x_1 - B_1x_1y_1 \\ \dot{y}_1 = -C_1y_1 + D_1x_1y_1 - E_1y_1z_1 \\ \dot{z}_1 = -F_1z_1 + G_1y_1z_1 \end{cases} \tag{1}$$

But even if the feeding is unlimited the density of predator's population can not grow infinitely, because of shortage of some their resources, for example a territory. A competition for such resources can be introduced in the third equation of (1) by means of member $-Hz^2$. Then we have:

$$\begin{cases} \dot{x}_1 = A_1x_1 - B_1x_1y_1 \\ \dot{y}_1 = -C_1y_1 + D_1x_1y_1 - E_1y_1z_1 \\ \dot{z}_1 = -F_1z_1 + G_1y_1z_1 - H_1z_1^2 \end{cases} \tag{2}$$

This system depends on eight parameters. Substituting the variables by

$$x_1 = k_1x, \qquad y_1 = k_2y, \qquad z_1 = k_3z, \qquad t_1 = k_4t \tag{3}$$

we change the scale of measure in order to eliminate four of the parameters.

Let us select the values of $k_i$ so that

$$A_1k_4 = 1; \qquad B_1k_2k_4 = 1; \qquad D_1k_1k_4 = 1; \qquad k_3k_4 = 1 \tag{4}$$

and mark

$$k_4C_1 = C; \qquad E_1k_3k_4 = E, \qquad F_1k_4 = F, \qquad G_1k_2k_3 = G \tag{5}$$

Then the system would look as

$$\begin{cases} \dot{x} = x - xy \\ \dot{y} = -Cy + xy - Eyz \\ \dot{z} = -Fz + Gyz - z^2 \end{cases} \tag{6}$$

marking $x_1 = x$, $x_2 = y$ and $x_3 = z$, we get

$$\begin{cases} \dot{x}_1 = x_1 - x_1x_2 = F(x_1, x_2, x_3) \\ \dot{x}_2 = -Cx_2 + x_1x_2 - Ex_3x_2 = G(x_1, x_2, x_3) \\ \dot{x}_3 = -Fx_3 + Gx_2x_3 - x_3^2 = H(x_1, x_2, x_3) \end{cases} \tag{7}$$

We can assert, considering the biological meaning of the coefficients, in the (2) that

$$\begin{cases} A_1 > 0, \quad B_1 > 0, \quad C_1 > 0, \quad D_1 > 0 \\ E_1 > 0, \quad F_1 > 0, \quad G_1 > 0, \quad H_1 > 0 \end{cases} \tag{8}$$

considering (3), (4) and (5) we get, that coefficients $k_i$, $C$, $E$ and $G$ are positive. Besides from the biological sense of solutions we are interested only in positive $x_i (i = 1 \div 3)$.

Let as find the equilibrium state of (7)system:

$$\begin{cases} \dot{x}_1 = 0 \\ \dot{x}_2 = 0 \\ \dot{x}_3 = 0 \end{cases} \Rightarrow \begin{cases} x_1(1 - x_2) = 0 \\ x_2(-C + x_1 - Ex_3) = 0 \\ x_3(-F + Gx_2 - x_3) = 0 \end{cases} \tag{9}$$

1. $x_1^0 = 0,\ x_2^0 = 0,\ x_3^0 = 0$, (point $A$)
2. $x_1^0 = 0,\ x_2^0 = 0,\ x_3^0 = -F$, (point $B$)
3. $x_1^0 = 0,\ x_2^0 = 0,\ x_3^0 = 0$, (point $C$)
4. $x_1^0 = 0,\ x_2^0 = \frac{F}{G} - \frac{C}{EG},\ x_3^0 = -\frac{C}{E}$, (point $D$)
5. $x_2^0 = 1,\ x_1^0 = C + Ex_3^0 = C + E(G - H),\ x_3^0 = -F + Gx_2^0 = -F + G$, (point $E$)

We assume that the quantity of each component is sufficiently large. This permits to disregard the influence of stochastic effects. Therefore the exploration of the balance points on the coordinate axes and planes is not interesting from the biological viewpoint.

## 3. The Exploration of the Positive Equilibrium State (point E)

Let us define the coefficients area so that from (9) we get that

$$\begin{cases} G - F > 0 \\ C + E(G - F) > 0 \end{cases} \tag{10}$$

There are the conditions of existing of nontrivial positive equilibrium state.

Linearizing (7) in the vicinity of (9) positive point, we get such a system:

$$\begin{cases} x_1 = -[C + E(G - F)](x_2 - x_2^0) \\ x_2 = (x_1 - x_1^0) - E(x_3 - x_3^0) \\ x_3 = G(G - F)(x_2 - x_2^0) - (G - F)(x_3 - x_3^0) \end{cases} \tag{11}$$

Characteristical polynomial of the system (11) is

$$\lambda^3 + \lambda^2(G - F) + \lambda[C + E(G - F) + EG(G - F)] + (G - F)[C + E(G - F)] = 0 \tag{12}$$

In order the stationary point of differential was equalities system stable, it would be necessary and sufficient that the real parts of roots of characteristical polynomial of linearized systems were negative.

But by the Rauss-Gurwitch theorem in order the roots of the polynomial $a_0\lambda^2 + a_1\lambda^2 + a_2\lambda + a_3 = 0$ were negative, the fulfillment of such inequalities would be necessary and sufficient:

$$\begin{cases} a_1 > 0 \\ \begin{vmatrix} a_1 & a_3 \\ a_0 & a_2 \end{vmatrix} > 0, \text{ i.e. } a_1a_2 - a_0a_3 > 0 \\ \begin{vmatrix} a_1 & a_3 & 0 \\ a_0 & a_2 & 0 \\ 0 & a_1 & a_3 \end{vmatrix} = a_3 \begin{vmatrix} a_1 & a_3 \\ a_0 & a_2 \end{vmatrix} > 0 \end{cases} \tag{13}$$

The last inequality is equivalent $a_3 > 0$, i.e. the conditions (13) can be written so:

$$\begin{cases} a_i > 0 \quad (i = 1, 1, 3) \\ a_1a_2 - a_3a_0 > 0 \end{cases} \tag{14}$$

For the polynomial (12) the conditions (14) are

$$\begin{cases} G-F>0 \\ C+E(G-F)>0 \\ (G-F)^2EG>0 \end{cases} \tag{15}$$

Comparing (15) with the conditions of existing nontrivial positive equilibrium state (10), we see that if for the system (7) exists the positive equilibrium state it would be always stable. But this state exists for every positive $G, C, E, F$ if only $G > F$. (If $G > F$ and $C > 0$, $E > 0$, then automatically $C + E(G - F) > 0$).

The fourdimensional parameter area

$$G>0, \quad C>0, \quad E>0, \quad F>0, \tag{16}$$

is divided by hyperplane $G = F$ into two areas with different solutions behaviour in each of them. There is no positive stationary solution in one of them and there is a stable solution in every point of another, i.e. the $E$ point is stable knot or focus.

## 4. Building of Liapunov's Vector-function

The (7) system is a particular case of such a system

$$\dot{N}_i = N_i(b_i - a_{ij}N_j), \quad i = 1 \div n \tag{17}$$

For this system

$$\|N\| = \|x_1, x_2, x_3\| \tag{18}$$

and matrix $\|B\|$ and $\|a_{ij}\|$ are the following:

$$\|b_i\| = \left\| \begin{matrix} 1 \\ -C \\ -F \end{matrix} \right\|, \qquad \|a_{ij}\| = \left\| \begin{matrix} 0 & 1 & 0 \\ -1 & 0 & E \\ 0 & -G & 1 \end{matrix} \right\| \tag{19}$$

We shall show that matrix $A = \|a_{ij}\|$ is positively $D$-dissipative.

Let matrix $D$ be

$$D = \left\| \begin{matrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{matrix} \right\|, \quad d_i > 0, \quad i = 1 \div 3$$

Then

$$DA = \left\| \begin{matrix} 0 & d_1 & 0 \\ -d_2 & 0 & d_2E \\ 0 & -a_2G & d_3 \end{matrix} \right\| \tag{20}$$

A quadratic form, corresponding to this matrix, is

$$F(x) = \|x_1, x_2, x_3\| DA \left\| \begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix} \right\| = x_1x_2(d_1 - d_2) + x_2x_3(d_2E - d_2G) + x_3^2 d_3 \tag{21}$$

Let it be

$$d_3 > 0, \quad d_1 - d_2 > 0, \quad d_2 E - d_3 G > 0 \tag{22}$$

We can always select $d_i$ so that the inequalities were fulfilled, e.g.:

$$d_3 = 1, \quad d_2 = \frac{2G}{E}, \quad d_1 = \frac{3G}{E}. \tag{23}$$

Then quadratic form (21) is positively defined, i.e. matrix is positively $D$-dissipative.

Besides that, system (7) has a positive equilibrium state in $R^3_+$ defined by equalities (9). Then from the theorem 2.12 from (Volltera, 1976) follows this function

$$E_d(x) = \sum_{i=1}^{3} d_i \int_{x_i^0}^{x_i} \frac{x_i^0 - x}{x}\, dx \tag{24}$$

is energetic to the systems (7) in $\mathrm{Int} R^3_+$.

Liapunov's function has form

$$E_d(x) = -\sum_{i=1}^{3} d_i (x_i - x_i^0 \ln x_i) + E_0(x), \tag{25}$$

where

$$E_0(x) = \sum_{i=1}^{3} d_i (x_i^0 - x_i^0 \ln x_i^0)$$

Here $x_i^0$ $(i = 1 \div 3)$ are defined by equalities (9), and $d_i$ $(i = 1 \div 3)$ – by equalities (23) for instance.

The derivative of this function taken in power of system (17) is:

$$\begin{aligned} \dot{E}_d(x) &= -\sum_{i=1}^{3} d_i \left(1 - \frac{x_i^0}{x_i}\right) x_i \left(b_i - \sum_{j=1}^{3} a_{ij} x_j\right) \\ &= \sum_{i=1}^{3} (x_i^0 - x_i) \sum_{j=1}^{3} d_i a_{ij} (x_j^0 - x_j), \end{aligned} \tag{26}$$

where $a_{ij}$ are defined by(19).

It is obvious that $\int_{x_i^0}^{x_i} \frac{x_i^0 - x}{x}\, dx < 0$ be $x_i < x_i^0$ or $x_i > x_i^0$, hence $E_d(x) < 0$.

Matrix $A$, as it is shown, is positively $D$-dissipative, i.e. there is such a matrix $D$ that $DA$ is positively defined. We have that $E_d(x) > 0$ for every $||x|| \neq ||x^0||$ and $||x|| \in \mathrm{Int} R^3_+$.

Besides as

$$\lim_{x \to 0} \int_{x_i^0}^{x_i} \frac{x_j^0 - x}{x} dx \to \infty \tag{27}$$

and

$$\lim_{x \to \infty} \int_{x_i^0}^{x_i} \frac{x_j^0 - x}{x} dx \to \infty \tag{28}$$

then $\lim_{x \to 0} E_d(x) = \infty$ and $\lim_{x \to \infty} E_d(x) = \infty$.

Hence it follows that positively defined stationary solution (9) is asymptotically stable on the whole in $\mathrm{Int}R_+^3$ (by Theorem 2.13 from Volterra, 1976). Therefore, we have the case of global stability, i.e., if a positively determined equilibrium state exists, then every solution of the system approaches to the stationary, i.e., $||x^{(t)}|| \to ||x^0||$, when $t \to \infty$.

Further let us try to determine if there are reserved trajectories in the final part of the first octant. Using Medvedev's criteria (Medvedev, 1970), lets take

$$M(x_1, x_2, x_3) = \frac{1}{x_1 x_2 x_3}, \quad \text{then}$$

$$\frac{\partial(\widehat{F}M)}{\partial x_1} + \frac{\partial(\widehat{G}M)}{\partial x_2} + \frac{\partial(\widehat{H}M)}{\partial x_3} = -\frac{1}{x, x_2} < 0 \quad \text{and} \quad x_1 = 0, \; x_2 = 0$$

are integral planes. Hence from the Medvedev's criteria follows that there are no reserved trajectories in the final part of $\mathrm{Int}R_+^3$ area.

## 5. System of Modelling

Here the concrete mathematical model of a ecosystem of a lake-cooler Drukshiai of a nuclear power station Ignalina according (Svirezev et al., 1978; Odum, 1979; Park et al., 1974) is considered.

The below given system (29) involves the following assumptions: the most limiting is one substance, the behaviour of components is determined by food resources, not by the inner state (age, rate of sexes, etc.); the intensity of the interaction is proportional to the production of the quantity of the interacting components and does not depend on the total quantity $S$, i.e., the most limiting substance in the lake.

$$
\begin{cases}
\dfrac{dx_i}{dt} =\dfrac{1}{S}\left(a_{in}x_ix_n - x_i \sum\limits_{j\in K_i}(1+e_{ij})a_{ij}x_j - g_ix_i^2 - d_ix_i\right), \quad i=\overline{1,m};\\
\dfrac{dx_i}{dt} =\dfrac{1}{S}\left(x_i\sum\limits_{j\in P_i} a_{ij}x_j - x_i\sum\limits_{j\in K_i}(1+e_{ij})a_{ij}x_j - g_ix_i^2\right.\\
\qquad\left. - d_ix_i - r_ix_i\right), \quad i=\overline{m+1,n-2};\\
\dfrac{dx_{n-1}}{dt} =\dfrac{1}{S}\left(\sum\limits_{j=1}^{n-2}(g_jx_j^2+d_jx_j) - x_{n-1}\sum\limits_{j-\in K_{n-1}} a_{n-1j}x_j\right.\\
\qquad\left. - \alpha x_{n-1} + \sum\limits_{i,j} e_{ij}a_{ij}x_ix_j\right);\\
\dfrac{dx_n}{dt} =\dfrac{1}{S}\left(\sum\limits_{j=m+1}^{n-2} r_jx_j + \alpha x_{n-1}) - x_n\sum\limits_{i=1}^{m} a_{ni}x_i\right),
\end{cases}
\tag{29}
$$

where

$x_i$ – components which unite populations according to the functional groups:

$x_1,\ldots,x_m$ – procedures (macrophits, phytoplankton),

$x_{m+1},\ldots,x_{n-2}$ – consumers (zooplankton, bentos, fish),

$x_{n_1}$ – detritus and bacteria,

$x_n$ – biogene (the most limiting substance).

All $x_i$ are estimated by the units of the limiting substance and $\sum x_i = S$.

$a_{ij}$ – coefficients of assimilation-utilization,

$e_{ij}$ – the part of food transformed to excrements,

$r_i$ – coefficient of respiration,

$g_i$ – coefficient of interspacial competition,

$d_i$ – coefficient of natural mortality,

$p_i$ – set of indices of the components which are used by $i$-th,

$K_i$ – set of indices of the components using the $i$-th,

$\alpha$ – coefficient of the decomposition of detritus,

matrix $\{a_{ij}\}$ is symmetrical, in matrix $\{e-ij\}:\ e_{ij}=o$, if $e_{ij}\neq 0$. All the coefficients of the system are periodical in time as they depend on temperature and light intensity.

The main difference between the energetic model (Krishev, 1970) and the given system is that the latter allows to find out the steady perennial oscillations of numerical solutions within a wide range of meanings of parameters that are independent of the initial data.

For the numerical realization of equations (29) is taken, the matrix of interaction is presented in Table 1. While calculating we have assumed linear dependence of the coefficients of interaction on light intensity and temperature which changed harmoniously within a one-year period.

The coefficient of mortality rate was found using the formula

$$d_i = \frac{c_i}{(T - T_{i\,\min})(T_{i\,\max} - T)}, \tag{30}$$

where $T$ is the temperature at a given moment, $T_{i\,\max}$ and $T_{i\,\min}$ are temperatures of the complete mortality of the $i$-th due to over-heat and over- freezing, $c_i$ is coefficient of mortality rate without regards to temperature.

**Table 1** Interaction of the components of the model

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ | $x_8$ | $x_9$ | $x_{10}$ | $x_{11}$ | $x_{12}$ | $x_{13}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $x_1$ | − | | | | | | | | | | | | + |
| $x_2$ | | − | | | | | | | | | | | + |
| $x_3$ | | | − | | | | | | | | | | + |
| $x_4$ | | | | − | | | − | | | | − | | + |
| $x_5$ | | | | | − | − | − | | | | − | | + |
| $x_6$ | | | | | + | − | − | − | − | + | − | + | |
| $x_7$ | | | | + | + | + | − | − | − | + | − | + | |
| $x_8$ | | | | | | + | + | − | − | | | | |
| $x_9$ | | | | | | − | − | | | − | | | |
| $x_{10}$ | | | | + | + | + | + | | | | − | | |
| $x_{11}$ | | | | | | − | − | | | | | 0 | 0 |
| $x_{12}$ | − | − | − | − | − | | | | | | | + | 0 |

+ is stimulation, − is suppression, 0 is neutrality; $x_1$ is aerial, $x_2$ is floating, $x_3$ is bottom dwelling macrophyts; $x_4$ is diatomic, $x_5$ is blue-greenish algae; $x_6$ is proto-zooplankton, $x_7$ is placid zooplankton; $x_8$ is predatory zooplankton; $x_9$ is fish; $x_{10}$ is bacterioplankton; $x_{11}$ is bentos; $x_{12}$ is detritus, $x_{13}$ is biogen.

The table does not present the accumulation of detritus due to die-out of the components, and biogen $x_{13}$ resulting from respiration.

In the present case observation of the "paradox of phytoplankton" without introducing (Dombrovsky et al., 1979) the "outer hormone system" is possible. In fact, three groups of macrophyts and two groups of phytoplankton are competing for the biogene $x_{13}$, however, this does not lead to the extinction of less adapted types: due to matter balance and periodical coefficients, the elements which live under less favorable conditions still survive, their quantity fluctuates below the mean annual level.

Some results of the numerical integration are presented in figures 1 and 2. Figure 1 illustrates the independence of behavioristic decisions of the initial data. It shows that three trajectories which start from three different points converge after a while. Changes in the behaviour of solution that take place when the characteristics of the environment change are presented in figure 2. The behaviour of the system during a seven year period is shown. Since the end of the second year till the beginning of the sixth one the temperature had been rising by $5^0 C$ per year. Light intensity and the amplitude of annual thermal oscillations did not change. The rise of temperature has led to the extinction of the predatory zooplankton.

## Conclusions

1. The generalized mathematical model of an ecosystem is proposed; the model is investigated on the basis of a quality theory of a differential equations.

Fig. 1. Independence of the behaviour of solutions of the initial data

(Macrophyts and phytoplankton)

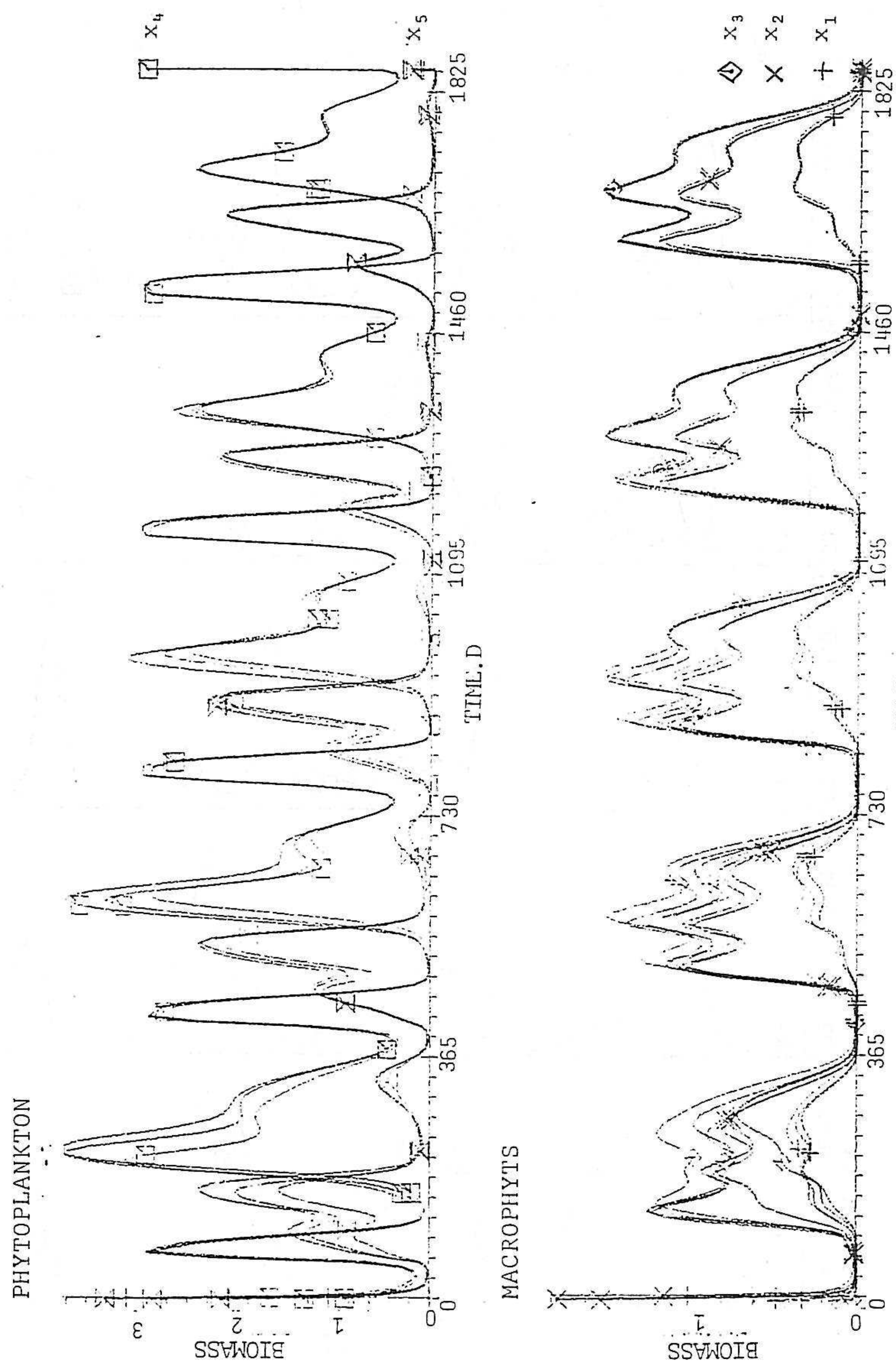
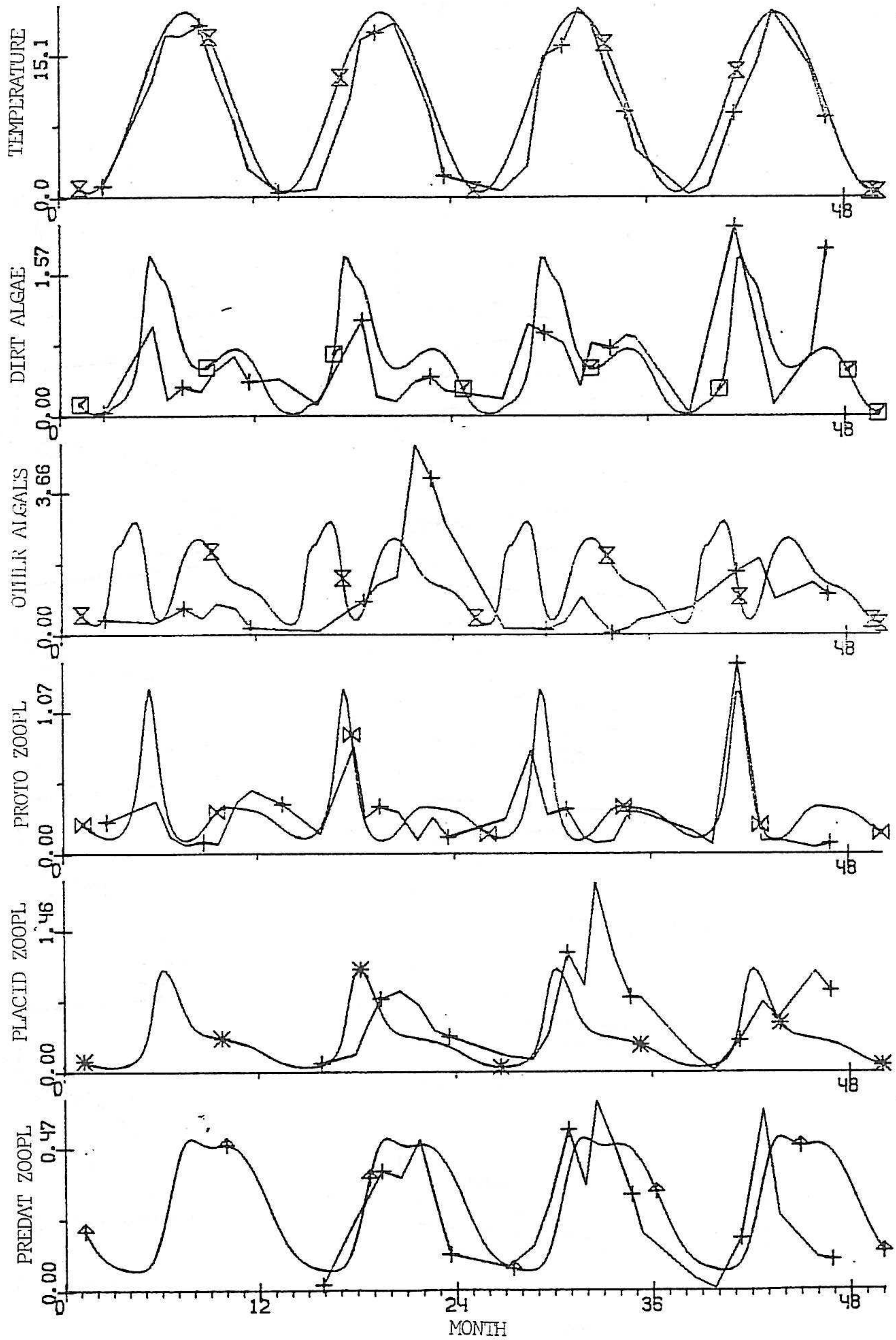


Fig. 2. The comparison of solution (x) with observation results (+)



2. The case of a global stability on the basis of the Liapunov's functions is proved, i.e. the existence of the positive equilibrium state, confirming that any decision of the system is approaches the stationary state.

3. The given mathematical model based on matter balance and first realized for the ecosystem of a lake-cooler shows the identity of the behaviour of solution to the actual processes of biocenosis.

4. The given model differs from the used energetical models in that it allows to get steady perennial oscillations of the numerical solutions within a wide range of changing parameters, which are independent on the initial data.

5. The results of simulation show the possibility to use the given model for the purposes of predicting the behavioristic characteristics of an ecosystem under thermal changes and other external factors.

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