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Political Life on a Lattice

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Abstract

Political Life on a Lattice: Toward an Elementary Theory of Nonlinear Politics

A political system is constructed using a probabilistic cellular automata model dubbed the Voter Model. We outline the formal properties of political life on the lattice as a collection of individuals where each individual's preference or attitude and the state of the system can be precisely known. On a lattice individuals are assumed to interact according to microscopic political interaction laws. These rules determine at every instant of time what the preference or attitude of each individual will be at the next instant of time, depending on the current preference or attitude of the other individuals. The core of the paper asks what the global, macroscopic properties of such a system of interacting individuals might be.

We simulate two political worlds using a stochastic Voter Model. One reflects a political reality in which decision makers have equally weighted preferences or attitudes which are held with equal salience. Power is uniformly distributed. We show that in finite time the outcome of specific interactions results in a macroscopic time-series which indicates deterministic chaos. The political universe defined by lattice models including elites diminishes the presence of chaos and allows long lived groups of voters to live in clusters. Within the rules of the stochastic Voter Model elites appear to stabilize influence. We speculate on the meaning of these simulations for political theory. With modifications, we suggest, it is possible to investigate how specific elites emerge, communicate globally, change their relative share of political power, move through time and space to gain political support, form coalitions, and eventually take control of complex political systems. With methods under development complex political change can be examined and analyzed for critical predictive conditions.

1. Introduction

Political science is concerned with explaining political things. Politics, no matter whether at the communal, state, national, or international level, is always the result of interactions among individuals. The individuals are all members of the respective political collective. While some individuals may be equipped with more power and attain an elite status, even elites are interdependent. There is no political process that would not be driven by a large number of direct or indirect interactions among individuals. Presidents bases their decisions on interactions with their advisors. Senators chat in the halls before key votes. Congressmen and congresswomen caucus. And this behavior is not fundamentally different from the way a typical voter bases her decision, say to support a Republican or a Democratic candidate in an election based on interactions with individuals in her environment.

Let us then, for the moment, accept this view that politics is the result of elementary interactions among a large number of individuals. Let us further assume that each individual interacts with other individuals according to the same rules and that these rules, which we will call microscopic interaction laws, are somehow known to us. These rules may be thought of as determining at every instant of time what the state (preference or attitude) of each individual will be at the next instant of time, depending on the current state of the other individuals. For simplicity we assume a one-bit rule, that is each individual can only be in one of two states, on or off, Democrat or Republican, pro-abortion or anti-abortion.

The system then is the collection of all individuals, and the state of the system at any instant of time is given by specifying each individual's state at that time. Depending on the rules of interaction, the time evolution of the system may be deterministic or stochastic. It is then natural to ask what the global, macroscopic properties of such a system of interacting individuals are. Some questions to be further elaborated here and in subsequent research are:

- How fast can the system, starting from some given initial state, evolve into a qualitatively different state? Are there initial states from which the development to a different state is catastrophic, so that some macroscopic variable (e.g. the fraction of Democratic, a policy space or geographic boundary between those pro and con) undergoes a rapid change?
- Are there initial states that evolve eventually into a state where all individuals are pro, i.e. into a uniform equilibrium state? Are there other equilibrium states which the system may approach after long times? Are there initial states from which the system never settles down into an equilibrium state?
- What is the generic behavior of the system at long times? Are there statistical properties and patterns (averages, fluctuations, boundaries between pro and con domains) that are independent of the details of the initial state and therefore characteristic of some microscopic political interaction law?
- Do some of these patterns persist under a modified microscopic interaction law and, thus, are universal for an whole class of interactions?
- Which macroscopic variables of the system are predictable in the sense that the variables' initial values uniquely determine the values at any later time, or allow of an estimation of later values with less than exponentially diverging error bounds? Which variables, by contrast, can be estimated only with exponentially diverging bounds and thus qualify for chaotic behavior in the sense of nonlinear dynamical systems? Can such variables lead the way to find some autonomous global dynamics involving chaotic attractors?
- Do the dynamics lead to emerging hierarchical structures which may correspond to political elites, alliances, or political units (made up of many individuals) that interact with each other similarly, but with some renormalization, to how the individuals interact?
- How does such a system compare with real political life, with data from specific political events, and with long-term observational data (time series) of various political variables within various political contexts?

While these questions are primordial for any understanding of complex political processes, the fact is that they have never been asked in political science and answers are nonexistent. The study of politics has lagged behind other disciplines in developing dynamical models beyond linear, special-purpose models. Examples for the latter are models of riots (Salert and Sprague, 1980), arms spirals, and short-term election forecasts. The deficiency has become particularly obvious and acute during recent political events in Europe, the Soviet Union, and the Middle East. These events and rapid macropolitical changes were as unexpected to any political scientist as the 1989 San Francisco earthquake came unpredicted by any geologist. Our point here is not that developments such as the ones in Eastern Europe and the Middle East could have been predicted in detail if better models had been available. The point is that despite the unpredictability of such developments there may be definite patterns in their occurrence which originate from how individuals interact at the microscopic level; there may be well-defined relations between the time scales on which short-term prediction is possible and the characteristics of chaotic evolution at long times; there may be indicators of incipient macropolitical changes; and there may be a wealth of macroscopic variables which are predictable at long times even though the system state is not.

Below we briefly describe typical properties of a lattice model (Sec. 2), and then discuss the relation of dynamical lattice models to earlier concepts and frameworks in political behavior (Sec. 3). Then we proceed to illustrate the dynamical properties of a stochastic version of one such model, the Voter Model, and how it is influenced by the presence of elites (Sec. 4). We conclude and give a forward look at work some of the work to come (Sec. 5).

2. Microscopic Dynamics: Lattice Simulations

2.1 Conceptual Framework

A lattice is a periodic d-dimensional array of discrete sites. Ideally the number of sites, N, is infinite. On the computer it will be a very large number, say $N \sim 10^6$. The lattice is a primitive structure analogous to discrete sample space. A site will be labelled by an index i or j as needed. The lattice may be square, triangular, or have yet some other geometric structure of the unit cell. On a square lattice, each lattice site has 2d nearest neighbors, i.e., immediately adjacent sites which share a common bond, and the size of the unit cell is characterized by the distance a between such adjacent sites (Fig. 1). Each site represents an individual decision maker which we call a voter. The voters are spatially arranged to reflect physical or geographic distance. Thus, voters are fixed in place: there is no migration. No other variables exist at this level of analysis. The lattice we consider here will always be a square lattice in two dimensions ($d = 2$) unless stated otherwise.

To lattice site i is associated an independent variable s_i which defines the state of the individual i ($i = 1, 2, \ldots, N$). We assume a binary state space where the variable s_i takes on one of two values. If $s_i = 0$ we say the site or voter is a Democrat (pro-choice, or ...).[1] If $s_i = 1$ we call the voter a Republican (anti-choice, etc.). Although individuals remain in fixed physical locations their partisanship, or issue position or ideology, may change as a result of interactions (to be specified below) with other voters on the lattice. At this conceptual stage there are no hierarchies and hence there are no elites, candidates, governments, nor interest groups. Without governments, candidates, or political parties, there are no elections. This is the most primordial political world possible.

2.2 Configurations

The collection of values $(s_1, s_2, \ldots, s_N)$ is called a configuration of the system, and defines the state of the entire system. There are 2^N states of the entire system. When we consider the configuration of the system at a particular time t, we will denote the state by $(s_1(t), s_2(t), \ldots, s_N(t))$. Having defined the state of the system, we can determine the properties of the system. Aggregated properties such as the number of Republican and Democratic sites, the average partisanship of a site ($(1/N)\ \Sigma_i\ s_i$), its variance may be readily evaluated. Any such aggregated property is of the form $F(s_1, s_2, \ldots, s_N)$ where F is some given function. Additional, spatial-temporal properties, including hierarchical aspects of the entire system as they may emerge, will be considered later on. Finally, in order to specify the lattice of voters as a dynamical system, we have to define the time evolution (dynamics). This is done by assuming that time is a discrete variable and by specifying how the current state of the system, at time t, determines the state at the next instant of time, t+1. A general time evolution is specified by

$$s_i(t+1) = G_i\ (t, s_1(t), \ldots, s_N(t)) \qquad (i = 1, \ldots, N) \qquad (1)$$

where G_i are given functions of time t and the current values $s1(t), \ldots, s_N(t)$.[2] Thus, starting from a given initial state

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[1] If a more refined description of the voter is desired, this can easily be accomplished by letting s_i take the additional value of 0.5 to designate a partisan independent, or by letting s_i take on a continuous range of values or even vector values.

[2] A more general model would be $s(t+1) = G\ (t, s(t), s(t-1), \ldots)$

where $s(t) \equiv \{\ s_i\ |\ i \in u\ \}$.

$(s_1(0), \ldots, s_N(0))$ at time $t = 0$, Eq. (1) by repeated application completely determines the state $(s_1(t), \ldots, s_N(t))$ at any later time t. While the G_i's may seem like an innocent collection of functions, it contains all possible "governing" systems, $(2^N)^N$, which in our simple binary world is $(2^{10^4})^{10^4}$. For instance, the assignment

$$s_i(t+1) = s_i(t) \qquad (i = 1, \ldots, N) \tag{2}$$

(i.e., $G_i(s_1, s_2, \ldots, s_N) = s_i$ for all i) would mean that every site retains its previous value regardless of the value at any other site. It amounts to a collection of noninteracting individuals and leads to a trivial time evolution. At the opposite end, the assignment

$$s_i(t+1) = s_{233}(t) \qquad (i = 1, \ldots, N) \tag{3}$$

(i.e., $G_i(s_1, s_2, \ldots, s_N) = s_{233}$ for all i) would mean that all sites become Democratic if site 233 is so, and become Republican if site 233 is Republican. It would model a strongly, but trivially interacting system of individuals who follow an influential at site 233 (action at a distance).

The state-evolution functions G_i refer to the preference of each individual. Since the function G_i describes how the state of one individual evolves as a result of its interaction with other individuals, we will refer to the G_i's also as interaction law. Eqs. (2) and (3) are examples for interaction laws that do not (explicitly) depend on time t. The time evolutions we are interested in assume interaction laws that are time-independent, uniform, and local.

Uniform means that G_i depends on i in the same way for all i's, that is contingent only on position on the lattice; local means that G_i depends only on more or less close neighbors of i. That is, the G_i is of the general form

$$G_i(s_1, \ldots, s_N) = g(s_i, s_{j1(i)}, \ldots, s_{jn(i)}) \tag{4}$$

where $j_{1(i)}, \ldots, j_{n(i)}$ identifies the n neighbors which influence i. The function g and the neighbor functions $j_1, \ldots, j_n$ completely determine the time evolution of every individual by Eqs. (1) and (4). In the following subsections we discuss some particular choices for neighborhoods, associated functions g, and their resulting properties for long-time behavior. The time evolution (1), defined on the state variables $s_i(t)$ on the lattice, is an example of a cellular automaton.

2.3 Model I: Deterministic Majority Law

The simplest example for an interaction law in which each individual interacts in the same way with its neighbors is the deterministic majority rule law. It makes each site adopt the value prevailing among the site and its 4 nearest neighbors on the lattice. The interaction law G_i for this electoral behavior is given by

$$G_i\,(s_i\,(t+1),\ldots, s_N\,(t+1)) = \begin{cases} 1 & \text{if} \quad \sum_{|i-j| \le a} s_j\,(t) \ge 3 \\ 0 & \text{else} \end{cases} \tag{5}$$

where

$$|i - j| = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2} \tag{6}$$

is the Euclidean distance between sites i and j, with location (x_i, y_i) and (x_j, y_j), respectively. The summation in (5) is over all sites j whose distance from i is less than or equal to spacing and, therefore, includes the site i and its 4 nearest neighbors (cf. Fig. 1). Thus, Eq. (5) says that the voter at site i will be Republican at time t + 1 if, among i and its 4 nearest neighbors, at least 3 are Republicans at time t; else it will be Democratic. Since the summation condition always examines 5 sites, no ties can occur and the interaction law is symmetrical with respect to partisan influence.

In terms of the function g introduced in Eq. (4), the majority law (5) can equivalently be expressed as

$$g(s_i\ ,\ s_{j1(i)}\ ,\dots,\ s_{j4(i)}) = \begin{cases} 1 \quad \mathit{if} & [\ s_i + s_{j1(i)} + \dots + s_{j4(i)}\ ] \geq 3 \\ 0 \quad \mathit{else} & \end{cases} \tag{7}$$

where $j_1(i)$, ..., $j_4(i)$ is i's neighbor to the North, ..., West, respectively. Eq. (7) manifests uniformity of the interaction law by the circumstance that the expression [...] depends on the state of site i and of i's neighbors in the same for all i; and it shows that the local neighborhood of any site i consists of i and its 4 nearest neighbors.

To illustrate how the time evolution defined by Eq. (5) operates, we consider the fate of a chosen site i, starting from some given initial state of i, its 4 nearest neighbors, and their respective nearest neighbors, over a sequence of two time steps. This is done in Fig. 2 in which we have chosen the initial state to be 0 for the "central" site i and its 4 nearest neighbors, and 1 for the 8 "peripheral" sites. In the first time step, the interaction of the 4 neighbors of i with the peripheral sites leads to a conversion of the 4 neighbors into Republicans, leaving i Democratic. In the second time step, also the central site i becomes Republican.

This illustrates how, in the course of time, a voter is affected by voters outside the range of the nearest-neighbor interaction law (5). Indeed, had we chosen the voter to the Northwest of i to be Democratic instead of Republican at t = 0, then i would have remained Democratic at t = 2. More generally, the state $s_i(t)$ will be influenced by the initial states $s_j(0)$ of all sites j for which the inequalities

$$|y_j - y_i + (x_j - x_i)| \leq ta \tag{8a}$$
$$|y_j - y_i - (x_j - x_i)| \leq ta \tag{8b}$$

hold (t = 0,1,2,...). Thus, despite the simplicity of the interaction law (5) and its local nature, the long-time behavior of the state of any given site i depends very sensitively on the initial configuration of the whole system.

2.3.1 Boundary Conditions

The long-time behavior raises the question how the majority law (5) is implemented at the boundary of the finite lattice, as we have it in any computer simulation. The answer is that, on a lattice consisting of $N = N' \cdot N'$ sites, one usually imposes periodic boundary conditions by identifying the points $(x_i + N'a,\ y_i)$ and $(x_i,\ y_i + N'a)$ with the point $(x_i,\ y_i)$. This makes the finite system free of boundaries and the time evolution, Eqs. (1) and (5), well-defined for all t. Alternatively, one may keep (5) at the boundary and restrict the summation to include only the actually existing nearest neighbors, which amounts to introducing a Democratic bias at the boundary because if i has only 3 or 2 nearest neighbors it is harder to satisfy the condition that the sum over $s_j(t)$ be ≥ 3. Or one may choose to replace the condition

$$\sum_{|i-j| \leq a} s_j\ (t) \geq 3 \tag{9}$$

in (5) by

$$\sum_{|i-j| \leq a} s_j\ (t) \geq 2 \tag{10}$$

when i is a boundary site, which weakens the Democratic bias at the boundary. For a sufficiently large lattice, however, such different choices of boundary conditions yield essentially the same results for the time evolution from a statistical viewpoint. In the sequel we therefore will assume periodic boundary conditions, for simplicity, unless stated otherwise. Another effect of the finiteness of the lattice in computer simulations is that it makes the time evolution nominally periodic. This is because a system with N sites can only be in one of 2^N different configurations and thus will have to be back in an earlier configuration after 2^N time steps at most. But for $N \sim 10^6$, as we will be interested in, 2^N is effectively infinite. That is, for most initial states, the time evolution accessible in a computer experiment will not show any such periodicity due to the finite size of the system but will resemble the evolution on an infinite lattice instead.

2.4 Variants of the Deterministic Majority Law

Fig. 3 gives an idea of what the time evolution on a large lattice looks like for an interaction law similar to (5). The interaction underlying Fig. 3 is the extended majority law in which site i, the 4 nearest neighbors of i, plus the 4 next-nearest neighbors of i are examined in order to determine the value of $s_{i(t+1)}$. Thus the extended majority law amounts to replacing (5) by

$$G_i\,(s_i\,(t+1)\,,\dots,\,s_N\,(t+1)) = \begin{cases} 1 & if \quad \sum_{|i-j| \le \sqrt{2}\,a} s_j\,(t) \ge 5 \\ 0 & else \end{cases} \tag{11}$$

Sometimes the neighborhood used in (5) is called a von Neumann neighborhood (nearest neighbors only), and the one used in (11) is called a Moore neighborhood (nearest and next-nearest neighbors). Fig. 3 provides an instance of where the initial configuration evolves quite rapidly into an essentially stationary configuration.

An interesting distinction between a von Neumann and Moore neighborhood definition is the likely occurrence of same-type sites within an initial random configuration. In Figure 3, the small black squares represent stable partisan (either Democrats or Republicans). The initial allocation of such stable partisan groups within a von Neumann neighborhood definition is approximately 35 percent, roughly 18 percent for each of the two partisan groups. In (5) governed by a uniform random assignment this initial allocation is likely to increase as the process of voting continues. Hence the asymptotic state of a von Neumann based voter rule is likely to maintain a sizable minority. A Moore neighborhood definition tends to homogeneity in that stable groups are not invasion proof.

2.5. Model III: Stochastic Majority Law

The Voter Model was first suggested by Clifford and Sudsbury (1973) and independently by Holley and Liggett (1975) and May and Martin (1975). The voter model converges to complete consensus in $d \leq 2$. On the surface, the process reflected by the Voter Model is trivial since at some time, t, the voters converge on a single opinion or belief, or the state does not change after t (Durrett, 1988). Yet with modifications and by not concentrating on the asymptotic results, the stochastic rule underlying the Voter Model we believe can reflect some interesting political attributes of individuals and give us insight into political systems in which such voters exist. The addition of a stochastic component to (5) produces the Voter Model (Liggett, 1985; Durrett, 1988).

$$G_i\,(s_i\,(t+1),\dots,\,s_N\,(t+1)) = \begin{cases} 1 & if \quad \Theta \quad \left(\sum_{|i-j| \le a} s_j\,(t) \ge 3 \right) \\ 0 & else \end{cases} \tag{12}$$

again where

$$|i-j| = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}$$

3.0 Dynamical Lattice Models and Political Behavior

3.1 Mass Electoral Behavior

Can political life be the product of simple, primitive rules of interaction? Cell space models have been used in physics (Heermann, 1990), chemistry and biology (Gutowitz, 1991; Forrest, 1991), and to a more limited degree in the economic and social sciences to investigate the ecological structure of behavior (Hayek, 1937; Haken 1975, 1983; Coucelius, 1985, 1986; Schelling 1971, 1978; Axelrod, 1986; Cowen and Miller, 1990).

Again, we assume that politics evolves from numerous, local interactions, too numerous to intuitively anticipate the outcome. By using "voting rules" on a lattice, enough interactions between decision makers can replicate interesting real political situations. We know that a highly ordered system evolves from what is essentially a random configuration (Clifford and Sudsbury, 1975) though the specific evolved configurations are themselves apparently random and are highly sensitive to initial conditions (Durrett, 1988). Thus when (12) is applied to a random initial configuration, these rules behave in a fashion which is completely dependent on the initial concentration of Democrats or Republicans. (See. Fig. 4 here)

Although it is known that a system is likely to converge to a homogeneous configuration under this rule of a Voter Model, it is not clear from prior research is the manner in which the convergence takes place. Figure 4 shows the long-term behavior of two separate runs of the Voter Model starting from very similar initial conditions. The upper trace starts from a randomly distributed population with each site having an initial probability of 0.50 of being assigned a 1 as the site value. The remainder are assigned 0. The lower trace is similar except that the initial probability of assignment of a value of 1 is 0.55.

The time paths of the two runs of the Voter Model could not be more different. Both converge on homogeneous values, but the outcomes are opposite. The upper trace, starting from an evenly matched distribution, converges on all 1's in the lattice, while the lower trace, with an excess of 1's in the lattice at the outset, converges on all zeros. A choice is made. The long-term behavior of the system approaches either all 1's or all 0's.

Figure 4 contains another feature of interest. Both of the traces are statistically self-similar since they scale according to a power law function (Feder 1988; Krassa and McBurnett 1991; Schroeder 1991). In fact, they closely resemble random walks, though Durrett (1988) proves that they are not. The initial analysis of the gross behavior of this system of individual interactions provides us with a number of puzzles: a well defined set of individual interactions, when considered in its aggregate behavior, follows an indeterminate path to its final destination.

Can the eventual outcome be determined in advance by the initial distribution of sites in the lattice? The simple stochastic Voter Model suggests a connection between the micro-level, the state of individual sites, and the macro-level, the configuration of 1's (or 0's). How does the aggregate time series converge on the equilibrium? Does this model converge in some orderly fashion that will allow prediction? Or, does this system behave in some erratic manner en route to its equilibrium? Indeed, can we divine the micro structure of this process from the aggregate output?

In this section we examine the dynamics of the upper trace. This trace converges on a homogeneous distribution of all 1's in the lattice. We begin by reconstructing the phase space using the time-delay method (Packard, et al. 1980; McBurnett, 1991). The reconstructed phase portrait is shown in Figure 5. To reconstruct the phase space with a single element time series a lagged value of the series is generated. The reconstruction of the phase space is accomplished by plotting the lagged value of the series against the present value of the series. In Figure 5, a 20 time step lag is used.

Figure 5 contains a pair of equilibria. One is in the lower left corner and the other is in the upper right corner of the figure. This strongly suggests that the aggregate dynamics from the Voter Model are not regular, nor will they be easy to describe. This phase portrait contains more than 2750 points, so the motion about the attractors can not be considered to be transient; the attractor in the lower left corner contains approximately 1000 points. This pattern is repeated for the time series not shown in phase space. The second series (the series which converges on all 0's) has a phase portrait that is very similar; it contains two equilibria and the orientation of the phase space is from SW to NE on the graph. The dynamics in the phase space suggest that the underlying attractor may be a strange attractor (Echmann and Ruelle, 1985).

The shape of attractors in low dimensional phase space can be recovered using the phase space representation from an appropriately lagged univariate time series (Packard et al., 1980; Nicolis and Prigogine, 1989). This result allows the use of the aggregate time series to analyze the dynamic properties of the time series. The dimension of the attractor can be measured in a number of ways. A measure that is computationally efficient for a discrete time series such as this one is called the correlation dimension (Nicolis and Prigogine, 1989). The correlation dimension provides a quantitative description of the rate of divergence of nearby points in the phase space. For many attractors, the correlation dimension is known to scale as a power law function. That is

$$C(r) \approx r^d \qquad (13)$$

or, in other words, the dimension of the attractor is determined by the slope of $\ln[C(r)]$ versus $\ln(r)$ in a particular range of r (Nicolis and Prigogine, 1989). This relation is

$$ln\ C(r) = d^* ln\ (r). \qquad (14)$$

The procedure then is to calculate C(r) for radii varying from 0 to that sufficient to cover all points in the phase space for successively higher dimensions. The relation between ln[C(r)] and ln(r) can be seen graphically in Figure 6 where the correlation integrals for dimensions 2 through 20 are plotted for the phase portrait of aggregate output from the Voter Model.

The slope d from equation 4 is estimated using bivariate regression from the various dimensions over the regions in Figure 6 that are approximately linear. It is the range of linearity that allows the inference of the dimension of the attractor. The slope is calculated for each dimension. If the slopes reach a saturation point, i.e., become parallel as the dimension increases, then the saturation point is interpreted to be the dimension of the attractor (Nicolis and Prigogine, 1989; Grassberger and Proccacia, 1983; Moon, 1987). Where the slopes fail to saturate and the correlation integral scales according to the relation

$$ln\ C(r) = d^* ln\ (n) \qquad (15)$$

where n is the dimension in which the correlation integral is constructed, the dynamical process is understood to be Gaussian white noise. This relationship is represented by a 45 line in the plane. This clearly differentiates the two processes that might be expected. Confirmation of the existence of the attractor is given by a series of estimates for d that lie under a 45 line.

It is not clear from Figure 6 which regions of the correlation integrals should be used to estimate the slopes in equation 4. We describe a technique that allows a straightforward graphical interpretation of the range of linearity for the various correlation integrals. The range of linearity is interpreted to be the portion of the correlation integrals which slope upward from the left and does not include the "flat" portion that gently slopes upward on the right part of the graph (when looking at the lower sets of points). Clearly there is some difference between the uppermost connected set of points, which is the correlation integral for dimension 2 and is very nearly flat over its entire range, and the lowest set of points, which is the correlation integral for dimension 20 and which contains a set of points with a steep positive slope and a set of points that is nearly flat in slope. We need to distinguish between these two qualitatively different structures.

To distinguish between them, we compute the first difference between adjacent points and plot the result. The abscissa on this graph is the log of the radius used in the prior calculation. This graph shows the change between adjacent points in Figure 7 for dimensions 10 through 20. By counting the number of adjacent points that exhibit change of some predetermined value, say greater than 0.01, one identifies the scaling region, the range of linearity over which the slope is estimated (Albano, et al., 1987).

Once the scaling region has been identified, the slope over that region is computed using OLS. We compute the slopes for dimensions 10-20 and display them, along with the standard error of each estimate, in Figure 8. Figure 8 shows that the slopes increase steadily in value up to dimension 16, after which the slopes "break" or saturate. Put another way, the slopes are (approximately) parallel after dimension 16.

Estimating the mean for the slope estimates for dimensions 16-20 and constructing a horizontal line that intersects the ordinate give a value 2.603. This represents the dimension of the attractor. Since this measure is noninteger, and the error bars for the estimates of the slopes over the scaling region fail to overlap any integer value, the attractor is a strange attractor and we conclude that the aggregate dynamics are chaotic for our elementary dynamic model of contextual interactions (Nicolis and Prigogine 1988; McBurnett 1991).

4. Elites in Political Structures

The effect elites have on the political process is debatable. The contention centers on the definition of elites, since elites exist at several levels and in multiple estates (Jennings and Miller, 1986).

We conceive of elites as opinion leaders who can influence others and induce their choice. Political elites therefore may act to induce voters to change their allegiance from one party to another or to modify a given opinion. Rules governing elite influence cannot by definition be either local or uniform. Under our current definition elites once activated, remain constant for the duration of the simulation. Elites are "hardwired" sites in the artifical political life that exists on a lattice. They represent an instance of global action-at-a-distance and non-uniform response to neighboring influences. Hence, if an elite is a nearest neighbor, in time, her influence can permeate a lattice and qualitatively influence its configuration.[3]

4.1 Elites on the Lattice

How many elites are there in a population? Eldersveld (1989) reports that elites "actually constitute less than one percent of the adult population of most communities." This provides us with a crude baseline for examining the potential influence of elites on a lattice structure. On a 50x50 lattice, we impose an elite by randomly placing sites on it that do not alter their opinion (1 or 0) according to the stochastic rule updating the sites on the lattice in the Voter Model. We assign 25 sites to fixed states. Sites are selected at random and assigned a value of 1 or 0 at random. Hence we have 12 elites assigned 1s and 13 elites assigned 0s (or vice versa) for each simulation. Again, eq. (12) governs interaction.

Figure 9 shows the aggregate output of a typical run of the modified Voter Model of 1000 iterations. While the Voter Model, with a finite number of lattice sites, in 2d is known to converge on a consensus in finite time, the imposition of elites on the lattice halts convergence. The time path of the aggregate distribution of sites shows an oscillatory pattern that continues through time.

The pattern which emerges with the inclusion of elites is qualitatively different from that which emerged without elites. We have shown that the dynamics for the stochastic Voter Model generates a chaotic time series. Another way to think about this process is to infer that aggregate opinion change in the lattice electorate is unpredictable in the absence of opinion leaders. While it may be clear from observation that the time series produced by the two model are different, the difference can be made more precise by examining the dynamics mathematically and contrasting the results.

4.1 Aggregate Dynamics with Elites

Two options are available. We could choose to examine the fractal structure of the output of the Voter Model with Elites, an analysis which we conducted with the Voter Model, by using the Correlation Integral over several dimensions, or we can examine the dynamical behavior in some other fashion. The dynamics of the two models (elite inclusive and no elites) appear to us to be qualitatively different. The first has been shown to have multiple equilibria and chaotic dynamics. Figure 10 appears to be quite regular with a stable oscillatory pattern emerging soon after the initiation of the process. To us, this time series does not appear to be chaotic and we choose a simpler (less time consuming) method to analyze these data and characterize its dynamic.

(Fig. 10 here)

We examine the Lyapunov exponent for this time series. Lyapunov exponents can take on three values which capture the long-term dynamical behavior of a time series in all its possibilities (Brown and McBurnett, 1992; Moon, 1988; Schuster, 1989). The exponent can be negative, zero, or positive. Negative or zero exponents are associated with stable dynamic processes. Positive exponents coincide with chaotic dynamic processes The technique we use to determine the Lyapunov exponent is the Wolf, et al. Fixed Evolution Time Algorithm (see Schaffer, et al. 1988).

Table 1 gives a set of values for Lyapunov exponents from a variety of initial conditions for the time series shown in Figure 10 and a comparative Lyapunov exponent for the Voter Model phase portrait shown in Figure 11 (Two attractor phase portrait).

Table 1

Lyapunov Exponents for the Voter Model with Elites

Run	Dimension	Time Evolution	Time Delay	Lyapunov Exp.
1	9	10	20	-0.0018
2	9	5	20	-0.0025
3	5	3	20	-0.0027
4	5	5	10	-0.0031
	Lyapunov Exponent for the Stochastic Voter Model			
5	5	100	20	+0.0102

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[3] In subsequent work we will design rules that allow for the emergence and time evolution of an elite. The purpose of this modification is to discover the effects elites can have on the dynamics of opinion as the Voter Model rules operate over long time scales.

The entries in this table show that this dynamical series is stable. All the Lyapunov exponents (in the upper portion of the table) are negative. The lower entry (run 5) shows the Lyapunov exponent for the stochastic Voter Model, and is positive. Hence the Voter Model, without elites, displays a chaotic time trajectory and sensitive dependence on initial conditions. With regard to the Voter Model with elites included, this table presents solid evidence that the system is not sensitive to the initial conditions. Over time, nearby initial values converge, rather than diverge. Under conditions of the stochastic voter model elites are stabilizing influence who influence opinion among voters. Indeed the lattices suggest that their influence is distributed through the lattice over time. This pattern matches the theoretical expectations and observable past patterns of elite behavior.

5. Discussion and Conclusion

The abstraction of this exercise may mask important theoretical implications. At a very cursory level a cellular voter has only two possible states: Democrat or Republican. Yet the voter can also viewed as a tenacious partisan, when the site survives an interaction with neighbors, or a converted partisan, where the site flips or converts based on the efforts of others. Modifying the rule would allow us to permit voters to be either nonaligned or independent nonpartisan (where for example, a cell site is empty), or a mobilized partisan (where the cell is a new partisan hatched from the set of the previous nonaligned, independent partisans after interactions with partisans). The resulting states based on efforts of neighbors reinforce the voter's current traits or convert the voter to a new position or identification. Similar labels could identify any cogent binary political trait. And of course, more complex vectors can be used instead of binary, scalar values for the cell traits. As such, the lattice design can provide the same complexity of design as any multivariate analysis.

On the surface, political behavior on a lattice appears to be very simple minded. The approach we take is equal to asserting that voters lack any serious choice mechanisms when to make political decisions. Voters instead are social learners who respond who respond uniformly to the nature of their own past traits and those of their defined nearest neighbors. Yet are these assumptions so drastically different from the traditional view of voters offered from nearly four decades of survey research (Campbell, et al., 1960; Fiorina, 1978)?

The voter model investigated here includes chaotic processes. This means that the dynamics at the system-level may prove intractable and unknowable, as the aggregate levels cannot be predicted for time $t+1$ based on aggregate measure such as the the fractions at time t. Even complete information at one level may not allow much insight at the other. In lattice models without elites, the two dynamics while inseparable and causally linked are also perhaps impossible to specify jointly. Because of chaos in the dynamic, the the two levels may be impossible to link even for any particular case will be available to cover all possible cases of the voter model.

We have been able to observe a world without political elites. Under normal circumstances of the Voter Model, a site has not jurisdiction over it's nearest or next nearest neighbors. That is, it simply cannot take over a cell, but must work collectively to alter its state. Hence, within most specifications of the Voter Model (either deterministic or stochastic), the outcome of specific interactions in time will result in either a homogeneous configuration or invariant (linear) blocks. This would reflect a political reality where all decision makers have equally weighted preferences or attitudes which are held with equal salience. It would be a world where political power is uniformly distributed.

The political universe defined defined by lattice models may find that elites may be long lived clusters of cell sites. Within the rules of the stochastic Voter Model elites appear to be stabilizing influences. Yet simple modifications may induce results where elites emerge and may eradicate former elites. In the language of lattice dynamics, if there is no coherent cluster pointing in one direction (or holding one set of traits), elites will not be well defined. With modifications it is possible to investigate how specific elites communicate, change their relative share of political power, move through time and space to gain political support, form coalitions, and eventually take control of complex political systems. The conditions of such political change can be examined and analyzed for critical predictive conditions.

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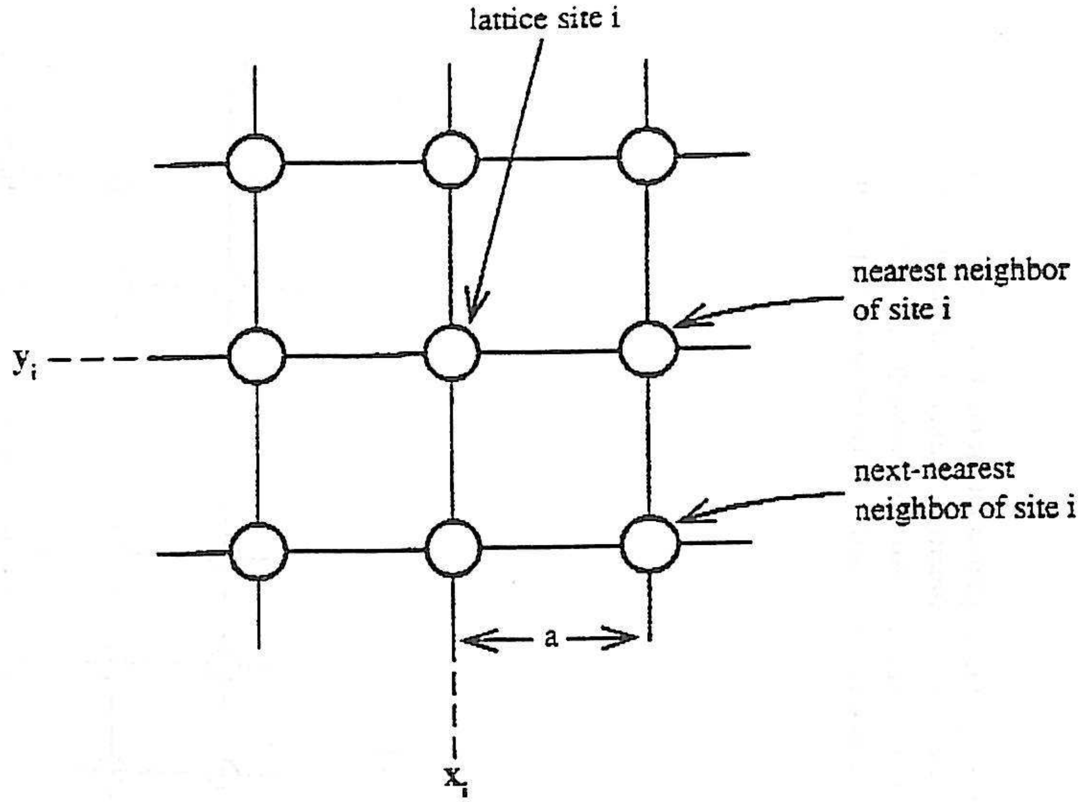
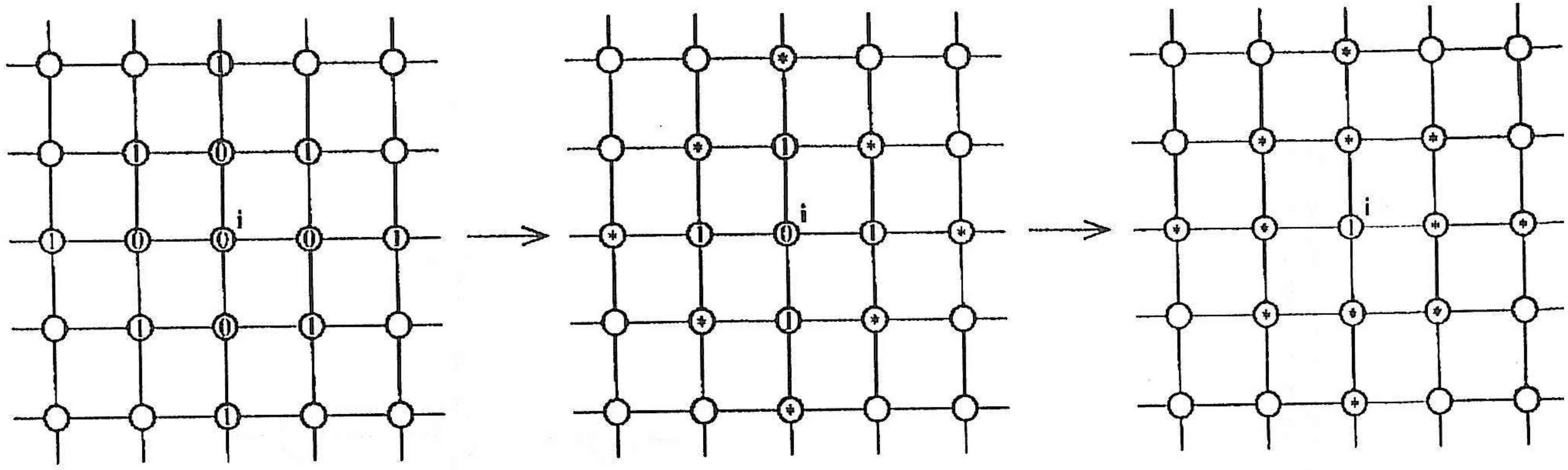


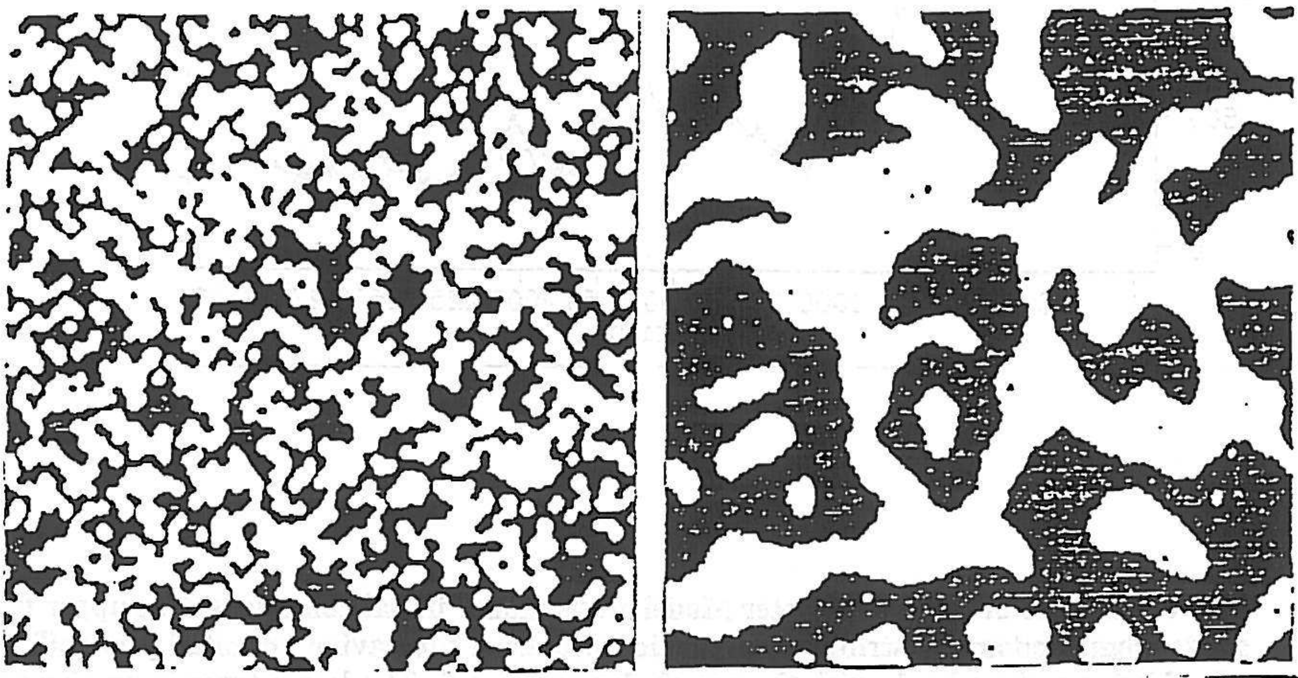
Fig. 1. A representative lattice site i and its neighbors on a square lattice with d = 2 and lattice constant a. The coordinates x_i and y_i designate the physical location of site i.



t = 0 t = 1 t = 2

Fig. 2. First two steps of the time evolution of a restricted segment of the lattice, centered at some site i, under the deterministic majority law. The values in the circles display the partisanship of the different sites; the symbol * means that the state cannot be updated without additional knowledge of the initial configuration beyond what is shown for t = 0.

Fig. 3. Time evolution of an initial configuration with 50% Republicans (black) and 50% Democrats (white) to a configuration at some later time



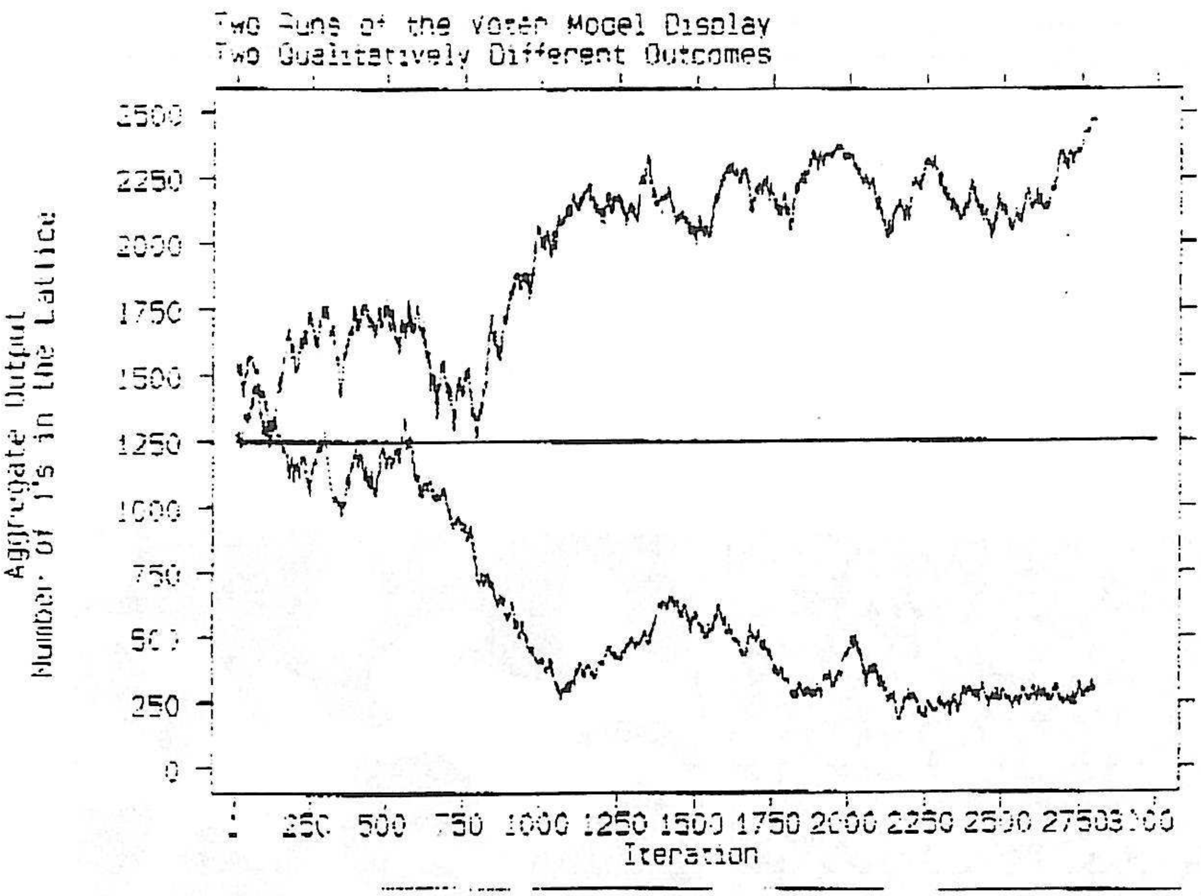


Fig. 4. Behavior of Stochastic Voter Model given similar initial conditions. The upper trace starts from randomly distributed population with each site having an initial probability of .50 of being assigned a 1, with the remainder assigned 0. The lower trace is similiar with the exception that the initial probability of being assigned a value of 1 is .55.

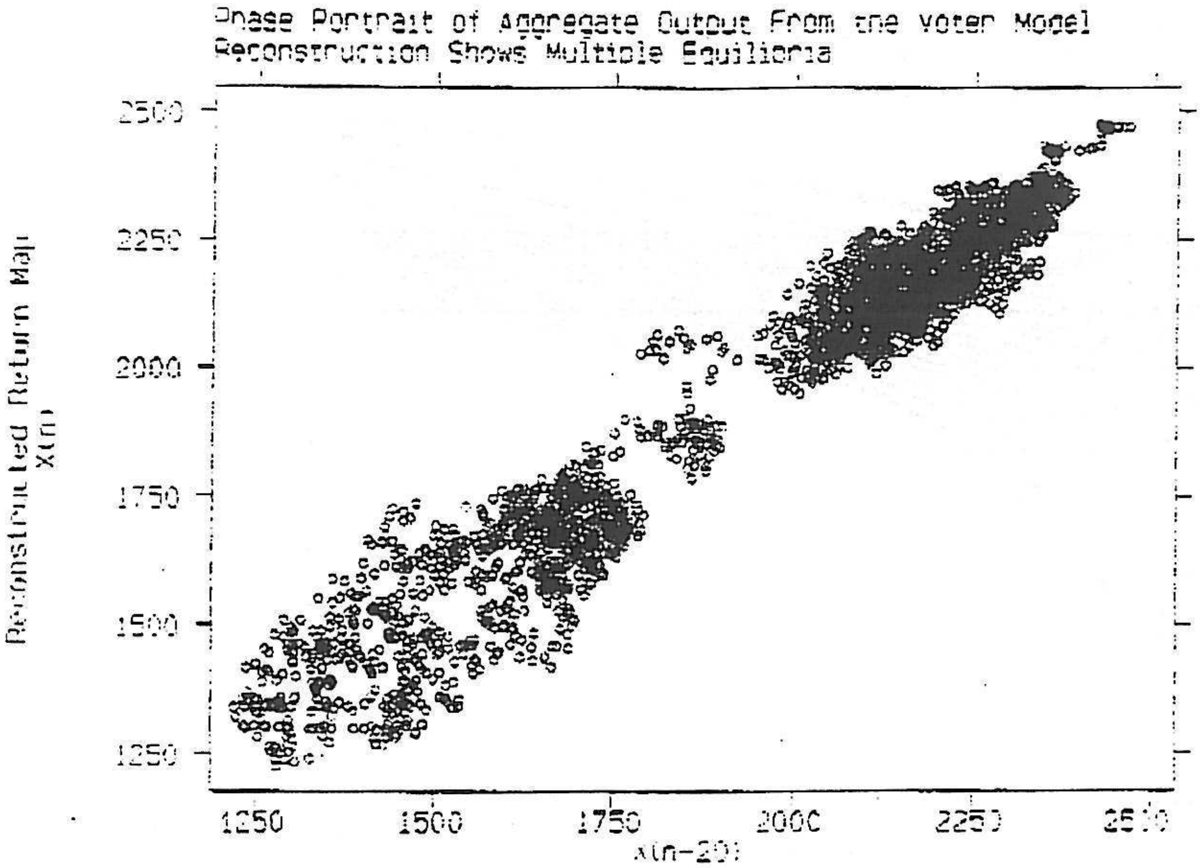


Fig. 5. Reconstructed phase portrait of the stochastic Voter Model. The present value of the series is plotted against the lagged value. A 20 time step lag is used.

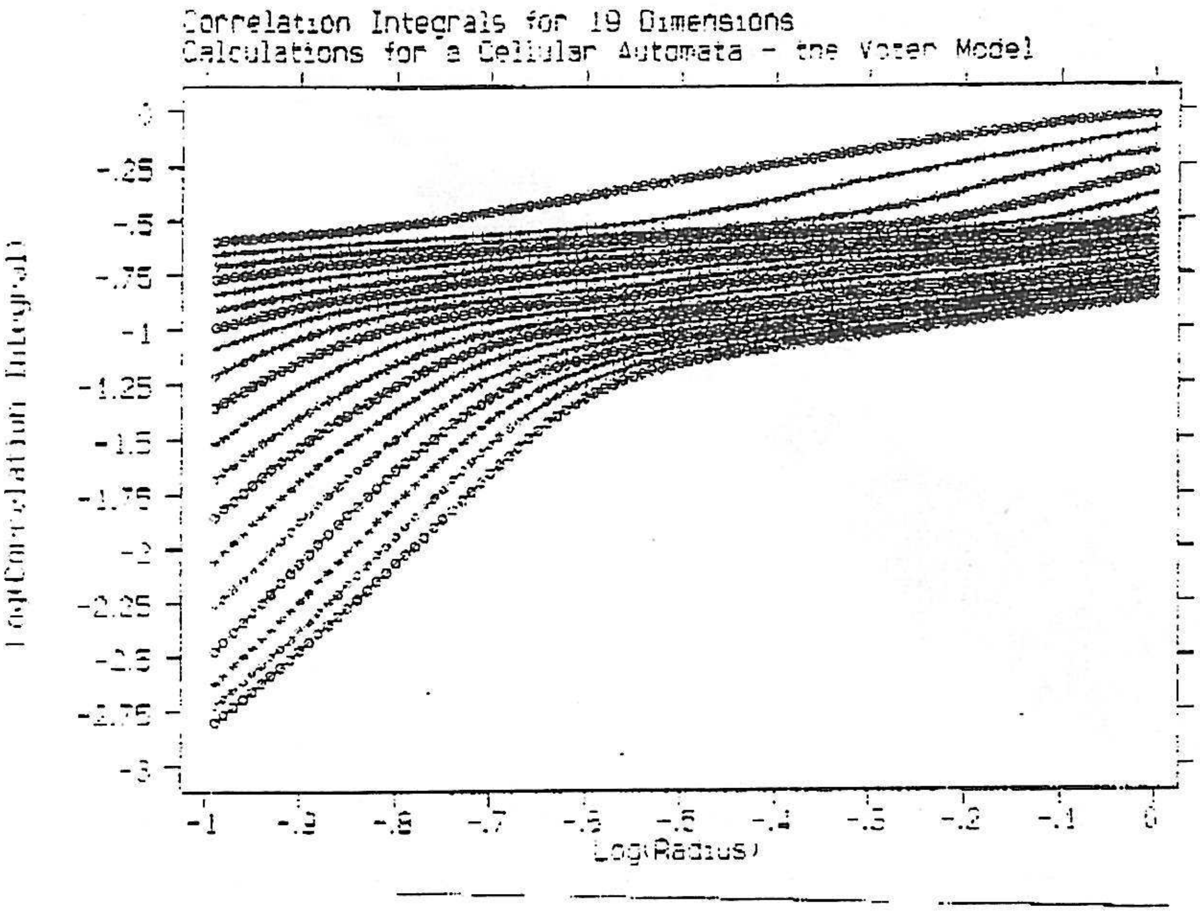


Fig. 6. Correlation integrals for dimensions 2 through 20 are plotted for phase portrait of aggregate output of stochastic Voter Model.

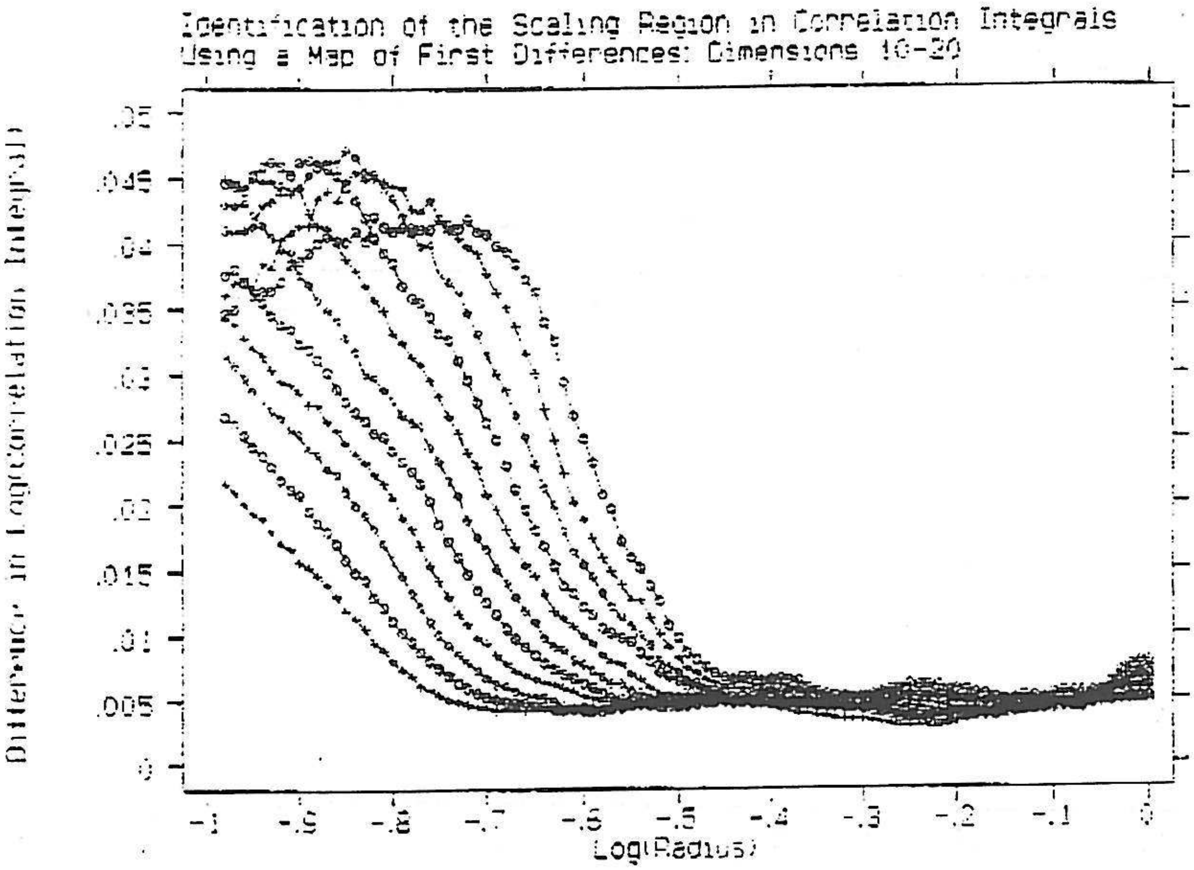


Fig. 7. Identification of Scaling Region for stochastic Voter Model in Correlation Integrals Using a Map of First Differences for Dimensions 10-20.

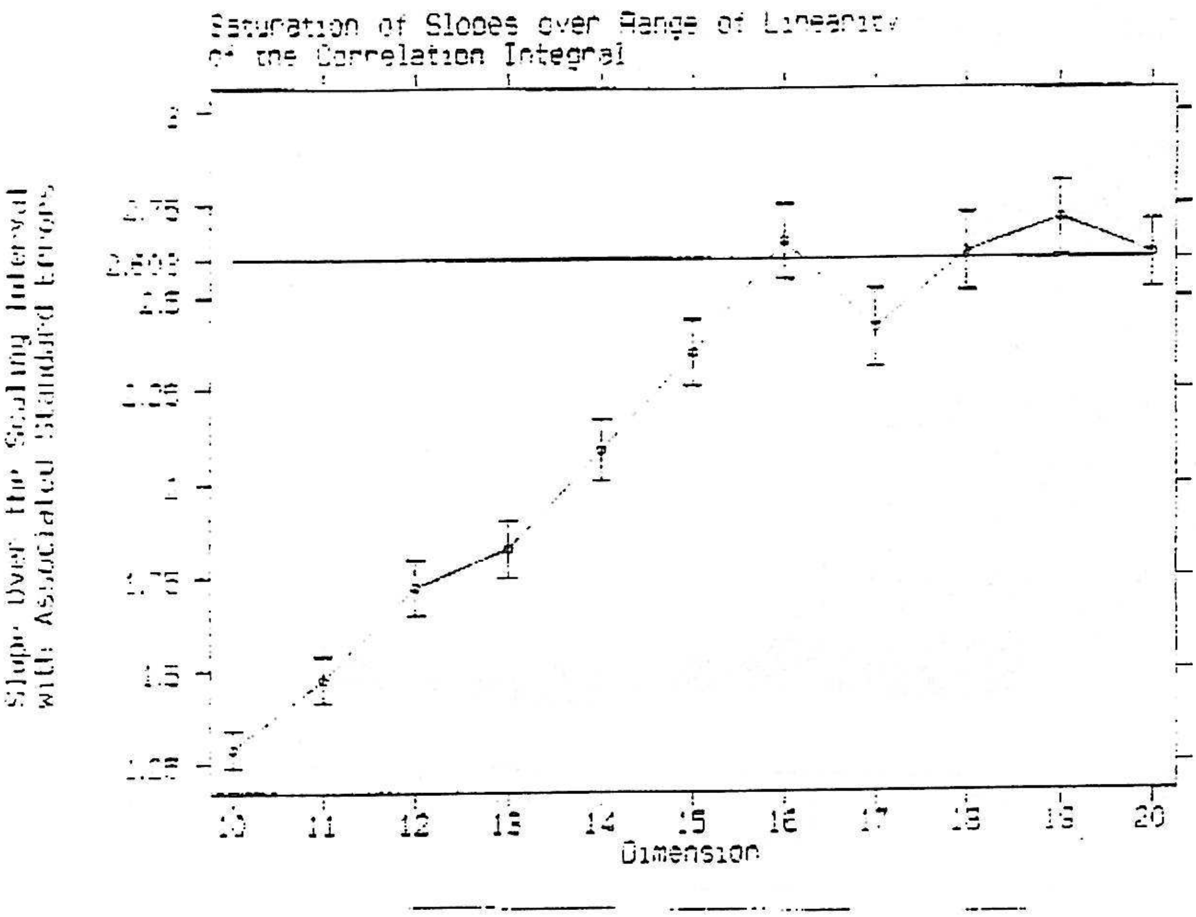


Fig. 8. Saturation of Slopes over Range of Linearity of the Correlation Integral.

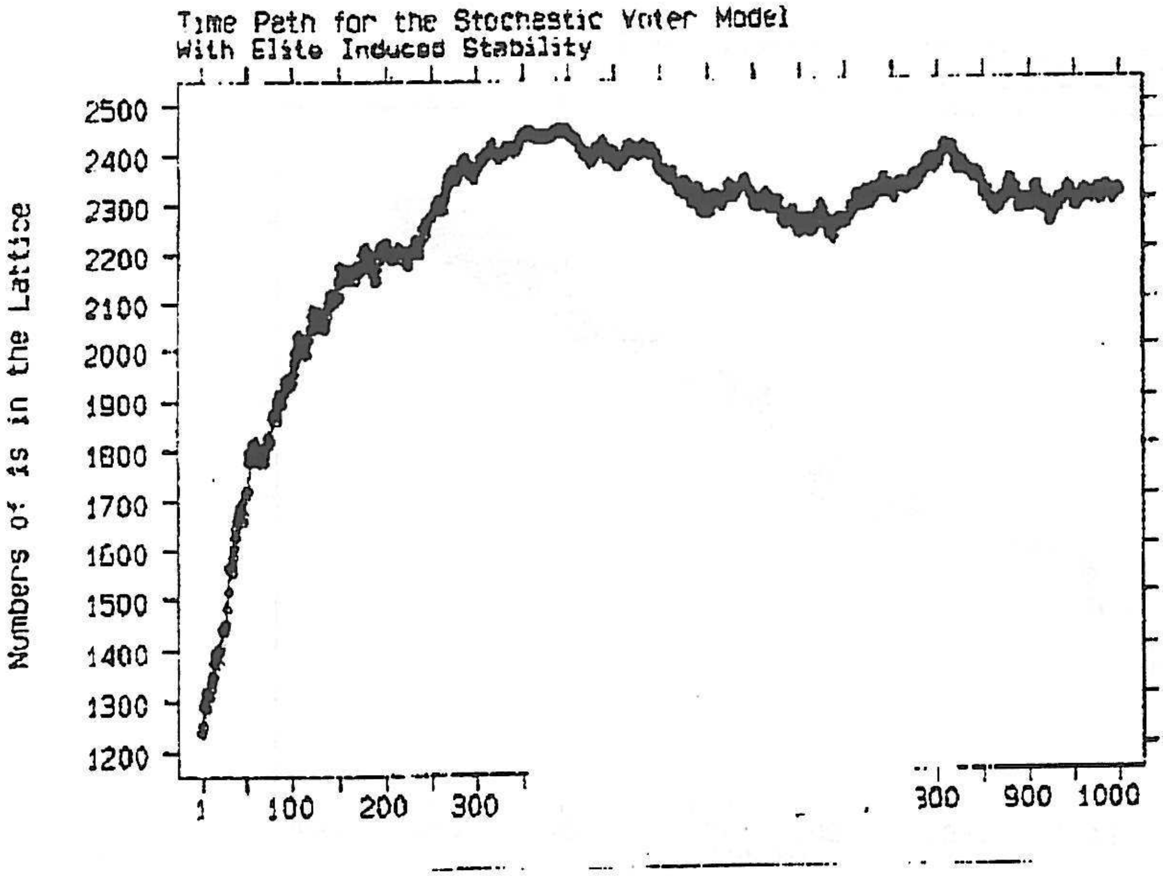


Fig. 9. Time Path for stochastic Voter Model with Inclusion of Elites.

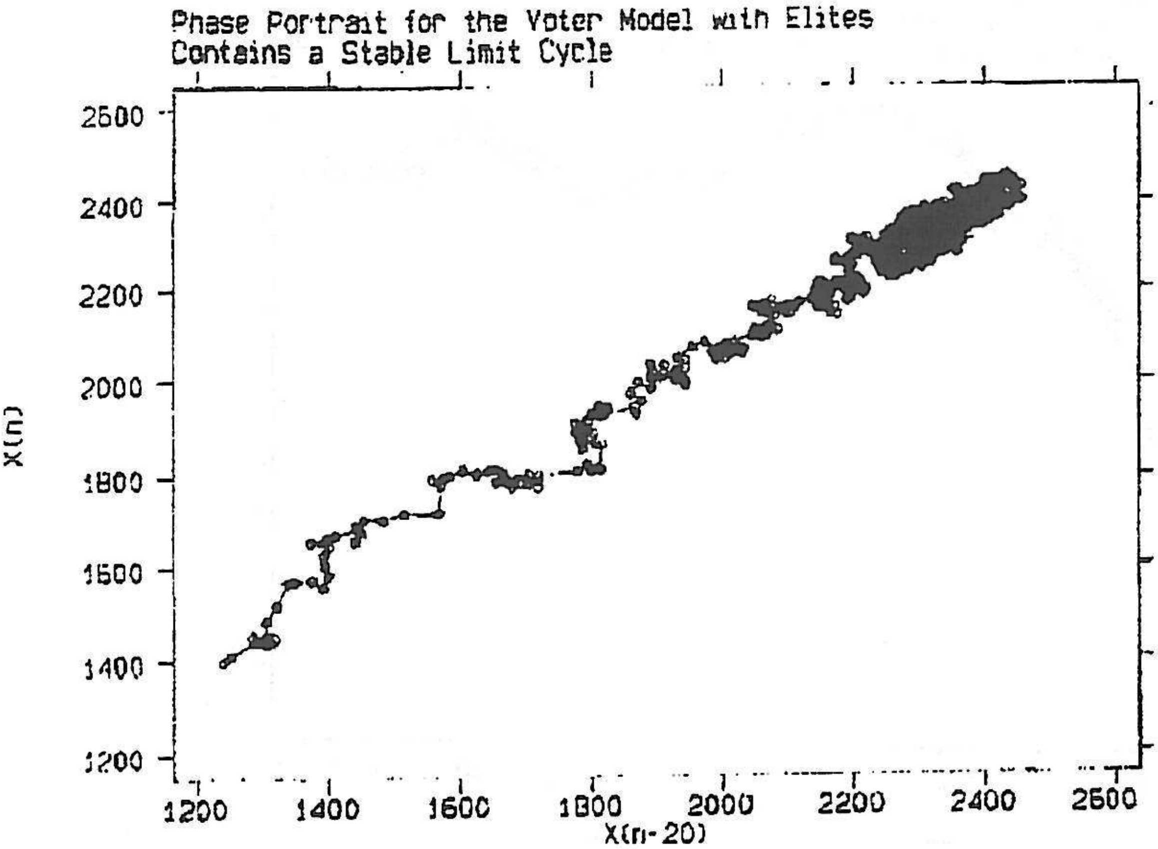


Fig. 10. Reconstructed phase portrait of the stochastic Voter Model with Inclusion of Elites. The present value of the series is plotted against the lagged value. A 20 time stop lag is used.